Generalized fuzzy rough sets based on fuzzy coverings

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Abstract. This paper further studies the fuzzy rough sets based on fuzzy coverings. We first present the notions of the lower and upper approximation operators based on fuzzy coverings and derive their basic properties. To facilitate the computation of fuzzy coverings for fuzzy covering rough sets, the concepts of fuzzy subcoverings, the reducible and intersectional elements, the union and intersection operations are provided and their properties are discussed in detail. Afterwards, we introduce the concepts of consistent functions and fuzzy covering mappings and provide a basic theoretical foundation for the communication between fuzzy covering information systems. In addition, the notion of homomorphisms is proposed to reveal the relationship between fuzzy covering information systems. We show how large-scale fuzzy covering information systems and dynamic fuzzy covering information systems can be converted into small-scale ones by means of homomorphisms. Finally, an illustrative example is employed to show that the attribute reduction can be simplified significantly by our proposed approach.

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Keywords: Rough set; Fuzzy covering; Information system; Homomorphism; Attribute reduction

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1 Introduction

Rough set theory, originally constructed on the basis of an equivalence relation, was proposed by Pawlak [17] for solving inexact or uncertain problems. But the condition of the equivalence relation is so restrictive that the applications of rough sets are limited in many practical problems. To deal with more complex data sets, many researchers have derived a large number of generalized models by replacing the equivalence relation with a few mathematical concepts such as fuzzy relations [1,5,6,14,19,20,22] and coverings [3,4,7,8,15,18,21,23,29,30,32–39] of the universe of discourse.

Recently, the theory of fuzzy rough sets has become a rapidly developing research area and got a lot of attention. For example, Dubois et al. [5,6] initially provided the rough fuzzy sets and the fuzzy rough sets. Then Radzikowska et al. [19,20] defined the fuzzy rough sets (respectively, the L-fuzzy rough sets) based on fuzzy similarity relations (respectively, residuated lattices). Afterwards, many researchers [4,7,15,33] investigated fuzzy rough sets based on fuzzy coverings. In practice, we need to compute the approximations of fuzzy sets in fuzzy covering approximation spaces. But the classical approximation operators based on coverings are incapable of computing the approximations of fuzzy sets in the fuzzy covering approximation space. It motivates us to extend approximation operators of covering approximation spaces for fuzzy covering approximation spaces. In addition, there are a large number of fuzzy coverings for the universal set in general. To facilitate the computation of fuzzy coverings for fuzzy covering rough sets, it is interesting to investigate the relationship among the elements of a fuzzy covering and operations on fuzzy coverings.

Meanwhile, many researches [9–12,16,24–28,40–42] have been conducted on homomorphisms between information systems with the aim of attribute reductions. For instance, Grzymala-Busse [10–12] initially introduced the concept of information system homomorphisms and investigated its basic properties. Then Li et al. [16] discussed invariant characters of information systems under some homomorphisms. Afterwards, Wang et al. [24] found that a complex massive covering information system could be compressed into a relatively small-scale one under the condition of a homomorphism, and their attribute reductions are equivalent to each other. Actually, we often deal with attribute reductions of large-scale fuzzy covering information systems in practical situations, and the work of Wang et al. mentioned above inspires that the attribute reduction of large-scale fuzzy covering information systems may be conducted by means of homomorphisms. But so far we have not seen any work on homomorphisms between fuzzy covering information systems. Additionally, the fuzzy covering information system varies with time due

to the dynamic characteristics of data collection, and the non-incremental approach to compressing the dynamic fuzzy covering information system is often very costly or even intractable. For this issue, we attempt to apply an incremental updating scheme to maintain the compression dynamically and avoid unnecessary computations by utilizing the compression of the original system.

The purpose of this paper is to investigate further fuzzy coverings based rough sets. First, we present the notions of the lower and upper approximation operators based on fuzzy coverings by extending Zhu's model [37], and examine their basic properties. Particularly, we find that the upper approximation based on neighborhoods can not be represented without using the neighborhoods as the classical covering approximation space [37] in the fuzzy approximation space. Second, we propose the concepts of fuzzy subcoverings, reducible and intersectional elements, union and intersection operations and investigate their basic properties in detail. Third, the theoretical foundation is established for the communication between fuzzy covering information systems. Concretely, we construct a consistent function by combining the fuzzy covering, proposed by Deng et al. [4], with the approach in [24], and explore its main properties known from the consistent function for the classical covering approximation space in [24]. We also provide the concepts of fuzzy covering mappings and study their basic properties in detail. Fourth, the notion of homomorphisms between fuzzy covering information systems is introduced for attribute reductions. We find that a large-scale fuzzy covering information system can be compressed into a relatively small-scale one, and attribute reductions of the original system and image system are equivalent to each other under the condition of a homomorphism. In addition, we give the algorithm to construct attribute reducts and employ an example to illustrate the efficiency of our approach for attribute reductions of fuzzy covering information systems. We also discuss how to compress the dynamic fuzzy covering information system.

The rest of this paper is organized as follows: Section 2 briefly reviews the basic concepts related to the covering information systems and fuzzy covering information systems. In Section 3, we put forward some concepts such as the neighborhood operators, the approximation operators and reducible elements for fuzzy covering approximation spaces, and investigate their basic properties. Section 4 is devoted to introducing the concept of consistent functions which provides the theoretical foundation for the communication between fuzzy covering information systems. In Section 5, we present the notion of homomorphisms between fuzzy covering information systems and discuss its basic properties. We also investigate data compressions of fuzzy covering information systems and dynamic fuzzy covering information sys-

tems. An example is given to illustrate that how to conduct attribute reductions of the fuzzy covering information system by means of homomorphisms. We conclude the paper and set further research directions in Section 6.

2 Preliminaries

In this section, we briefly recall some basic concepts related to the covering information system and fuzzy covering information system. Three examples are given to illustrate two types of covering information systems.

Definition 2.1 *[2] Let U be a non-empty set (the universe of discourse). A non-empty sub-family $\mathscr{C} \subseteq \mathscr{P}(U)$ is called a covering of U if*

(1) *every element in $\mathscr{C}$ is non-empty;*

(2) $\bigcup\{C|C \in \mathscr{C}\} = U$*, where $\mathscr{P}(U)$ is the powerset of U.*

It is clear that the concept of a covering is an extension of the notion of a partition. In what follows, $(U, \mathscr{C})$ is called a classical covering approximation space.

To investigate further coverings based rough sets, Chen et al. proposed the following concepts on coverings.

Definition 2.2 *[3] Let $\mathscr{C}$=$\{C_1, C_2, ..., C_N\}$ be a covering of U, C_x=$\bigcap\{C_i|x \in C_i$ and $C_i \in \mathscr{C}\}$ for any $x \in U$, and $Cov(\mathscr{C})$=$\{C_x|x \in U\}$. Then $Cov(\mathscr{C})$ is called the induced covering of $\mathscr{C}$.*

Suppose c is an attribute, the domain of c is $\{c_1, c_2, ..., c_N\}$, C_i means the set of objects in U taking a certain attribute value c_i, and $C_x = C_i \cap C_j$, it implies that the possible value of x regarding the attribute c is c_i or c_j, and C_x is the minimal set containing x in $Cov(\mathscr{C})$.

Definition 2.3 *[3] Let Δ=$\{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$ be a family of coverings of U, Δ_x=$\bigcap\{C_{ix}|C_{ix} \in \mathscr{C}_i, 1 \le i \le m\}$ for any $x \in U$, and $Cov(\Delta)$=$\{\Delta_x|x \in U\}$. Then $Cov(\Delta)$ is called the induced covering of Δ.*

That is to say, Δ_x is the intersection of all the elements including x of each $\mathscr{C}_i$, and it is the minimal set including x in $Cov(\Delta)$. Furthermore, $Cov(\Delta)$ is a partition if every covering in Δ is a partition. In what follows, (U, Δ) is called a covering information system. To illustrate how covering information systems are constructed, we present two examples which have different application backgrounds.

Example 2.4 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ be eight houses, $C = \{price, color\}$ the attribute set, the domains of *price* and *color* are $\{high, middle, low\}$ and $\{good, bad\}$, respectively. To evaluate these houses, specialists A and B are employed and their evaluation reports are shown as follows:

$$\begin{aligned} high_A &= \{x_1, x_4, x_5, x_7\}, middle_A = \{x_2, x_8\}, low_A = \{x_3, x_6\}; \\ high_B &= \{x_1, x_2, x_4, x_7, x_8\}, middle_B = \{x_5\}, low_B = \{x_3, x_6\}; \\ good_A &= \{x_1, x_2, x_3, x_6\}, bad_A = \{x_4, x_5, x_7, x_8\}; \\ good_B &= \{x_1, x_2, x_3, x_5\}, bad_B = \{x_4, x_6, x_7, x_8\}, \end{aligned}$$

where $high_A$ denotes the houses belonging to high price by the specialist A. The meanings of other symbols are similar. Since their evaluations are of equal importance, we should consider all their advice. Consequently, we obtain the following results:

$$\begin{aligned} high_{A\vee B} &= high_A \cup high_B = \{x_1, x_2, x_4, x_5, x_7, x_8\}; \\ middle_{A\vee B} &= middle_A \cup middle_B = \{x_2, x_5, x_8\}; \\ low_{A\vee B} &= low_A \cup low_B = \{x_3, x_6\}; \\ good_{A\vee B} &= good_A \cup good_B = \{x_1, x_2, x_3, x_5, x_6\}; \\ bad_{A\vee B} &= bad_A \cup bad_B = \{x_4, x_5, x_6, x_7, x_8\}. \end{aligned}$$

Based on the above statement, we derive a covering information system (U, Δ), where $\Delta = \{\mathscr{C}_{price}, \mathscr{C}_{color}\}$, $\mathscr{C}_{price} = \{high_{A\vee B}, middle_{A\vee B}, low_{A\vee B}\}$ and $\mathscr{C}_{color} = \{good_{A\vee B}, bad_{A\vee B}\}$.

Example 2.5 Let Table 1 be an incomplete information system, where $*$ stands for the lost value. According to the interpretation in [11], the lost value is considered to be similar to any value in the domain of the corresponding attribute. Consequently, we obtain three coverings of U by the attribute set as follows: $\mathscr{C}_{structure} = \{\{x_1, x_2, x_4, x_6\}, \{x_2, x_3, x_5, x_6\}\}$, $\mathscr{C}_{color} = \{\{x_1, x_2, x_5\}, \{x_3, x_4, x_5, x_6\}\}$, $\mathscr{C}_{price} = \{\{x_1, x_4, x_5, x_6\}, \{x_2, x_3, x_4, x_6\}\}$. Hence, (U, Δ) is a covering information system, where $\Delta = \{\mathscr{C}_{structure}, \mathscr{C}_{color}, \mathscr{C}_{price}\}$.

To conduct the communication between covering information systems, Wang et al. provided the concept of consistent functions based on coverings.

Definition 2.6 *[24] Let f be a mapping from U_1 to U_2, $\mathscr{C}=\{C_1, C_2, ..., C_N\}$ a covering of U_1, $C_x=\bigcap\{C_i | x \in$*

C_i and $C_i \in \mathscr{C}$}, and $[x]_f = \{y \in U_1 | f(x) = f(y)\}$. If $[x]_f \subseteq C_x$ for any $x \in U_1$, then f is called a consistent function with respect to $\mathscr{C}$.

Based on Definition 2.6, Wang et al. constructed a homomorphism between a complex massive covering information system and a relatively small-scale covering information system. It has also been proved that their attribute reductions are equivalent to each other under the condition of a homomorphism. Hence, the notion of the consistent function provides the foundation for the communication between covering information systems.

In order to deal with uncertainty and more complex problems, Zadeh [31] proposed the theory of fuzzy sets by extending the classical set theory. Let U be a non-empty universe of discourse, a fuzzy set of U is a mapping $A : U \longrightarrow [0,1]$. We denote by $\mathscr{F}(U)$ the set of all fuzzy sets of U. For any $A, B \in \mathscr{F}(U)$, we say that $A \subseteq B$ if $A(x) \leq B(x)$ for any $x \in U$. The union of A and B, denoted as $A \cup B$, is defined by $(A \cup B)(x) = A(x) \vee B(x)$ for any $x \in U$, and the intersection of A and B, denoted as $A \cap B$, is defined by $(A \cap B)(x) = A(x) \wedge B(x)$ for any $x \in U$. The complement of A, denoted as $-A$, is defined by $(-A)(x) = 1 - A(x)$ for any $x \in U$. Furthermore, a fuzzy relation on U is a mapping $R : U \times U \longrightarrow [0,1]$. We denote by $\mathscr{F}(U \times U)$ the set of all fuzzy relations on U.

In practical situations, there exist a lot of fuzzy information systems as a generalization of crisp information systems, and the investigations of fuzzy information systems have powerful prospects in applications. To conduct the communication between fuzzy information systems, Wang et al. proposed a consistent function with respect to a fuzzy relation.

Definition 2.7 *[27] Let U_1 and U_2 be two universes, f a mapping from U_1 to U_2, $R \in \mathscr{F}(U_1 \times U_1)$, $[x]_f = \{y \in U_1 | f(x) = f(y)\}$, and $\{[x]_f | x \in U_1\}$ is a partition on U_1. For any $x, y \in U_1$, if $R(u,v) = R(s,t)$ for any two pairs $(u,v),(s,t) \in [x]_f \times [y]_f$, then f is said to be consistent with respect to R.*

Based on the consistent function, Wang et al. constructed a homomorphism between a large-scale fuzzy information system and a relatively small-scale fuzzy information system. It has been proved that their attribute reductions are equivalent to each other under the condition of a homomorphism. In this sense, the notion of the consistent function provides an approach to studying the communication between fuzzy information systems.

Recently, Deng et al. [4] proposed the concept of a fuzzy covering.

Definition 2.8 *[4] A fuzzy covering of U is a collection of fuzzy sets $\mathscr{C}^* \subseteq \mathscr{F}(U)$ which satisfies*

(1) *every fuzzy set $C^* \in \mathscr{C}^*$ is non-null, i.e., $C^* \neq \emptyset$;*

(2) $\forall x \in U, \bigvee_{C^* \in \mathscr{C}^*} C^*(x) > 0$.

Unless stated otherwise, U is a finite universe, and $\mathscr{C}^*$ consists of finite number of sets in this work. In what follows, $(U, \mathscr{C}^*)$ is called a fuzzy covering approximation space, and (U, Δ^*) is called a fuzzy covering information system, where $\Delta^* = \{\mathscr{C}_i^* | 1 \leq i \leq m\}$. Throughout the paper, we denote the set of all fuzzy coverings of U as $C(U)$ for simplicity.

In the following, we employ an example to illustrate the fuzzy covering information system.

Example 2.9 Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ be eight houses, $C = \{price, color\}$ the attribute set, the domains of *price* and *color* are $\{high, middle, low\}$ and $\{good, bad\}$, respectively. To evaluate these houses, specialists A and B are employed and their evaluation reports are shown as follows:

$$\begin{aligned}
high_A^* &= \frac{1}{x_1} + \frac{0.7}{x_2} + \frac{0}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{0}{x_6} + \frac{1}{x_7} + \frac{0.65}{x_8};\\
middle_A^* &= \frac{0.6}{x_1} + \frac{1}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.45}{x_5} + \frac{0.5}{x_6} + \frac{0.5}{x_7} + \frac{1}{x_8};\\
low_A^* &= \frac{0}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} + \frac{0}{x_4} + \frac{0.5}{x_5} + \frac{1}{x_6} + \frac{0}{x_7} + \frac{0.5}{x_8};\\
good_A^* &= \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{1}{x_6} + \frac{0}{x_7} + \frac{0}{x_8};\\
bad_A^* &= \frac{0}{x_1} + \frac{0.4}{x_2} + \frac{0}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{0.2}{x_6} + \frac{1}{x_7} + \frac{1}{x_8};\\
high_B &= \frac{0.9}{x_1} + \frac{0.7}{x_2} + \frac{0}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{0}{x_6} + \frac{1}{x_7} + \frac{0.8}{x_8};\\
middle_B^* &= \frac{0.6}{x_1} + \frac{1}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.45}{x_5} + \frac{0.7}{x_6} + \frac{0.5}{x_7} + \frac{1}{x_8};\\
low_B^* &= \frac{0}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} + \frac{0}{x_4} + \frac{0.5}{x_5} + \frac{0.9}{x_6} + \frac{0}{x_7} + \frac{0.5}{x_8};\\
good_B^* &= \frac{0.8}{x_1} + \frac{1}{x_2} + \frac{0.9}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{1}{x_6} + \frac{0}{x_7} + \frac{0}{x_8};\\
bad_B^* &= \frac{0}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{0.2}{x_6} + \frac{0.9}{x_7} + \frac{1}{x_8},
\end{aligned}$$

where $high_A$ is the membership degree of each house belonging to the high price by the specialist A. The meanings of the other symbols are similar. Based on the above results, we obtain a fuzzy covering informa-

tion system (U, Δ^*), where $\Delta^* = \{\mathscr{C}^*_{price}, \mathscr{C}^*_{color}\}$, $\mathscr{C}^*_{price} = \{C_{high}, C_{middle}, C_{low}\}$ and $\mathscr{C}^*_{color} = \{C_{good}, C_{bad}\}$.

$$\begin{aligned}
C_{high} &= high^*_A \cup high^*_B = \frac{1}{x_1} + \frac{0.7}{x_2} + \frac{0}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{0}{x_6} + \frac{1}{x_7} + \frac{0.8}{x_8};\\
C_{middle} &= middle^*_A \cup middle^*_B = \frac{0.6}{x_1} + \frac{1}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.45}{x_5} + \frac{0.7}{x_6} + \frac{0.5}{x_7} + \frac{1}{x_8};\\
C_{low} &= low^*_A \cup low^*_B = \frac{0}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} + \frac{0}{x_4} + \frac{0.5}{x_5} + \frac{1}{x_6} + \frac{0}{x_7} + \frac{0.5}{x_8};\\
C_{good} &= good^*_A \cup good^*_B = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{1}{x_6} + \frac{0}{x_7} + \frac{0}{x_8};\\
C_{bad} &= bad^*_A \cup bad^*_B = \frac{0}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} + \frac{0.2}{x_6} + \frac{1}{x_7} + \frac{1}{x_8}.
\end{aligned}$$

It is obvious that we can construct a fuzzy covering of the universe with an attribute. Since the fuzzy covering rough set theory is effective to handle uncertain information, the investigation of this theory becomes an important task in rough set theory.

3 The basic properties of the fuzzy covering approximation space

In this section, we introduce the concepts of neighborhoods, the lower and upper approximation operators to facilitate the computation of fuzzy sets for fuzzy covering approximation spaces. Then we propose the concepts of fuzzy subcoverings, irreducible and reducible elements, non-intersectional and intersectional elements of fuzzy coverings. Afterwards, the union and intersection operations on two fuzzy coverings are provided. We also construct two roughness measures and employ several examples to illustrate the proposed notions.

3.1 The lower and upper approximation operations

Before introducing approximation operators, we present the concepts of neighborhoods and induced fuzzy coverings based on fuzzy coverings.

Definition 3.1 *Let $(U, \mathscr{C}^*)$ be a fuzzy covering approximation space, and $x \in U$. Then $C^*_{\mathscr{C}^* x} = \bigcap\{C^* | C^*(x) > 0$ and $C^* \in \mathscr{C}^*\}$ is called the neighborhood of x concerning $\mathscr{C}^*$.*

We notice that C^*_x is the intersection of all fuzzy subsets whose membership degrees of $x \in U$ are not zeroes. Assume that $C^*_1, C^*_2 \in \mathscr{C}^*$, $C^*_1(x) > 0, C^*_2(x) > 0$, and $C^*_x = C^*_1 \cap C^*_2$ for $x \in U$, it implies that the membership degree of x in C^*_x is $min\{C^*_1(x), C^*_2(x)\}$. In addition, we observe that the classical

neighborhood of a point $C_x = \bigcap\{C|x \in C \in \mathscr{C}\}$ is the same as that in Definition 3.1 if the membership degree for any $x \in U$ has its value only from the set $\{0, 1\}$, where $\mathscr{C}$ is a covering of U. For convenience, we denote C^*, $\mathscr{C}^*$, $C^*_{\mathscr{C}_i x}$ and $C_{\mathscr{C} x}$ as C, $\mathscr{C}$, C_{ix} and C_x, respectively.

We present the properties of the neighborhood operator below.

Proposition 3.2 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, and $x, y \in U$. If $C_x(y) > 0$, then $C_y \subseteq C_x$.*

Proof. Assume that $\mathscr{A} = \{C|C \in \mathscr{C}, C(x) > 0\}$ and $\mathscr{B} = \{C'|C' \in \mathscr{C}, C'(y) > 0\}$. Since $C_x(y) > 0$, it follows that $C(y) > 0$ for any $C \in \mathscr{A}$. Consequently, $C \in \mathscr{B}$. It implies that $\{C|C \in \mathscr{C}, C(x) > 0\} \subseteq \{C'|C' \in \mathscr{C}, C'(y) > 0\}$. Therefore, $C_y \subseteq C_x$. □

Proposition 3.3 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, and $x, y \in U$. If $C_x(y) > 0$ and $C_y(x) > 0$, then $C_y = C_x$.*

Proof. Straightforward from Proposition 3.2. □

Based on Definition 3.1, we present the concept of a fuzzy covering induced by the original fuzzy covering.

Definition 3.4 *Let $\mathscr{C}=\{C_1, C_2, ..., C_N\}$ be a fuzzy covering of U, $C_x=\bigcap\{C_i|C_i(x) > 0$ and $C_i \in \mathscr{C}\}$ for any $x \in U$, and $Cov(\mathscr{C})=\{C_x|x \in U\}$. Then $Cov(\mathscr{C})$ is called the induced fuzzy covering of $\mathscr{C}$.*

It is clear that C_x has the minimal membership degree of x in $Cov(\mathscr{C})$, and each element of $Cov(\mathscr{C})$ can not be represented as the union of other elements of $Cov(\mathscr{C})$. In other words, C_x is the minimal set containing x in $Cov(\mathscr{C})$. Furthermore, $Cov(\mathscr{C})$ is a fuzzy covering of U, and it is easy to prove that the concept presented in Definition 2.2 is a special case of Definition 3.4 when the values of membership degree are taken from the set $\{0, 1\}$.

An example is employed to illustrate the induced fuzzy covering.

Example 3.5 Let $U_1 = \{x_1, x_2, x_3, x_4\}$, and $\mathscr{C}_1 = \{C'_1, C'_2, C'_3\}$, where $C'_1 = \frac{1}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} + \frac{0.5}{x_4}$, $C'_2 = \frac{0.5}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.6}{x_4}$, and $C'_3 = \frac{0}{x_1} + \frac{0.5}{x_2} + \frac{0}{x_3} + \frac{0.5}{x_4}$. By Definition 3.4, we obtain the induced fuzzy covering $Cov(\mathscr{C}_1) = \{C_{x_i}|i = 1, 2, 3, 4\}$, where $C_{x_1} = C_{x_3} = \frac{0.5}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$, and $C_{x_2} = C_{x_4} = \frac{0}{x_1} + \frac{0.5}{x_2} + \frac{0}{x_3} + \frac{0.5}{x_4}$.

For convenience, we denote C'_i as C_i in the following examples.

We also propose the notion of a fuzzy covering induced by a family of fuzzy coverings.

Definition 3.6 *Let* $\Delta=\{\mathscr{C}_1,\mathscr{C}_2,...,\mathscr{C}_m\}$ *be a family of fuzzy coverings of* U*,* $\Delta_x=\bigcap\{C_{ix}|C_{ix} \in Cov(\mathscr{C}_i), 1 \leq i \leq m\}$ *for any* $x \in U$*, and* $Cov(\Delta)=\{\Delta_x|x \in U\}$*. Then* $Cov(\Delta)$ *is called the induced fuzzy covering of* Δ*.*

In other words, Δ_x is the intersection of all the elements whose membership degrees of x are not zeroes in each $\mathscr{C}_i$, and it is the set whose membership degree of x is the minimal in $Cov(\Delta)$. Furthermore, given $x, y \in U$, if $\Delta_x(y) > 0$, then $\Delta_y \subseteq \Delta_x$. Consequently, $\Delta_x(y) > 0$ and $\Delta_y(x) > 0$ imply that $\Delta_x = \Delta_y$. In addition, $Cov(\Delta)$ is a fuzzy covering of U. Therefore, it is easy to verify that the notion given in Definition 2.3 is a special case of Definition 3.6 when the values of membership degree are taken from the set $\{0, 1\}$.

Next, we give an example to illustrate Definition 3.6.

Example 3.7 Let $U_1 = \{x_1, x_2, x_3, x_4\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3\}$, $\mathscr{C}_1 = \{C_4, C_5, C_6\}$, $\mathscr{C}_2 = \{C_7, C_8, C_9\}$, and $\mathscr{C}_3 = \{C_{10}, C_{11}, C_{12}\}$, where $C_4 = \frac{1}{x_1} + \frac{1}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$, $C_5 = \frac{0.5}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4}$, $C_6 = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$, $C_7 = \frac{0}{x_1} + \frac{0}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}$ $C_8 = \frac{1}{x_1} + \frac{1}{x_2} + \frac{0.7}{x_3} + \frac{0.7}{x_4}$, $C_9 = \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$, $C_{10} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}$ $C_{11} = \frac{0.5}{x_1} + \frac{0.5}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}$, and $C_{12} = \frac{0.8}{x_1} + \frac{0.8}{x_2} + \frac{0.7}{x_3} + \frac{0.7}{x_4}$. By Definition 3.6, we obtain that $Cov(\Delta) = \{\Delta_{x_i}|i = 1, 2, 3, 4\}$, where $\Delta_{x_1} = \Delta_{x_2} = \frac{0.5}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$, and $\Delta_{x_3} = \Delta_{x_4} = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$.

In practice, the classical approximation operators based on coverings are not fit for computing the approximations of fuzzy sets in the fuzzy covering approximation space. To solve this issue, we propose the concepts of the lower and upper approximation operators based on fuzzy coverings by extending approximation operators in [37].

Definition 3.8 *Let* $(U, \mathscr{C})$ *be a fuzzy covering approximation space, and* $X \subseteq U$*. Then the lower and upper approximations of* X *are defined as*

$$\begin{aligned}
\underline{X}_{\mathscr{C}} &= \bigcup\{C|C \subseteq X \text{ and } C \in \mathscr{C}\};\\
\overline{X}_{\mathscr{C}} &= \Big(\bigcup\{C_x|X(x) > 0 \text{ and } \underline{X}_{\mathscr{C}}(x) = 0, x \in U\}\Big) \cup \underline{X}_{\mathscr{C}}.
\end{aligned}$$

The physical meaning of the lower and upper approximations of X is that we can approximate X by $\underline{X}_{\mathscr{C}}$ and $\overline{X}_{\mathscr{C}}$. Particularly, if $\overline{X}_{\mathscr{C}} = \underline{X}_{\mathscr{C}} = X$, then X can be understood as a definable set. Otherwise, X is undefinable. It is clear that the lower and upper approximation operations are the same as those [37] in the classical covering approximation space if $\mathscr{C}$ is a covering of U. In this sense, the notions given in Definition 3.8 are generalizations of the classical ones into the fuzzy setting. In the following, we investigate their basic properties in detail.

Proposition 3.9 *Let* $(U, \mathscr{C})$ *be a fuzzy covering approximation space, and* $X, Y \subseteq U$. *Then*

(1) $\overline{\emptyset}_{\mathscr{C}} = \emptyset$, $\underline{\emptyset}_{\mathscr{C}} = \emptyset$;

(2) $\underline{U}_{\mathscr{C}} \subseteq U$, $\overline{U}_{\mathscr{C}} \subseteq U$;

(3) $\underline{X}_{\mathscr{C}} \subseteq \overline{X}_{\mathscr{C}}$, $\underline{X}_{\mathscr{C}} \subseteq X$;

(4) $\overline{X}_{\mathscr{C}} \cup \overline{Y}_{\mathscr{C}} \subseteq \overline{(X \cup Y)}_{\mathscr{C}}$;

(5) $X \subseteq Y \Longrightarrow \underline{X}_{\mathscr{C}} \subseteq \underline{Y}_{\mathscr{C}}, \overline{X}_{\mathscr{C}} \subseteq \overline{Y}_{\mathscr{C}}$;

(6) $\forall C \in \mathscr{C}$, $\underline{C} = C, \overline{C} = C$;

(7) $\underline{(\underline{X}_{\mathscr{C}})}_{\mathscr{C}} = \underline{X}_{\mathscr{C}}$, $\overline{X}_{\mathscr{C}} = \overline{(\overline{X}_{\mathscr{C}})}_{\mathscr{C}}$;

(8) $\overline{(\underline{X}_{\mathscr{C}})}_{\mathscr{C}} = \underline{X}_{\mathscr{C}}$, $\underline{(\overline{X}_{\mathscr{C}})}_{\mathscr{C}} \subseteq \overline{X}_{\mathscr{C}}$.

Proof. Straightforward from Definition 3.8. □

Proposition 3.10 *The following properties do not hold generally in the fuzzy covering approximation space:*

(1) $\underline{(X \cap Y)}_{\mathscr{C}} = \underline{X}_{\mathscr{C}} \cap \underline{Y}_{\mathscr{C}}$;

(2) $\underline{(-X)}_{\mathscr{C}} = -(\overline{X}_{\mathscr{C}})$;

(3) $\overline{(-X)}_{\mathscr{C}} = -(\underline{X}_{\mathscr{C}})$;

(4) $\underline{(-\underline{X}_{\mathscr{C}})}_{\mathscr{C}} = -(\underline{X}_{\mathscr{C}})$;

(5) $\overline{(-\overline{X}_{\mathscr{C}})}_{\mathscr{C}} = -(\overline{X}_{\mathscr{C}})$;

(6) $\underline{U}_{\mathscr{C}} = U$, $\overline{U}_{\mathscr{C}} = U$;

(7) $\overline{(X \cup Y)}_{\mathscr{C}} \subseteq \overline{X}_{\mathscr{C}} \cup \overline{Y}_{\mathscr{C}}$;

(8) $X \subseteq \overline{X}_{\mathscr{C}}$.

Example 2 in [35] can illustrate that Proposition 3.10(1-5) does not hold generally in the fuzzy covering approximation space. Specially, we obtain that $\underline{U}_{\mathscr{C}} = U$ and $X \subseteq \overline{X}_{\mathscr{C}}$ do not necessarily hold for any $X \subseteq U$. Consequently, the lower and upper approximation operations are not interior and closure operators, respectively, in the fuzzy covering approximation space.

We employ an example to illustrate that Proposition 3.10(6-8) does not hold generally.

Example 3.11 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C} = \{C_{13}, C_{14}, C_{15}, C_{16}\}$, where $C_{13} = \frac{0.3}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0}{x_4}$, $C_{14} = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$, $C_{15} = \frac{0.3}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0.4}{x_4}$, and $C_{16} = \frac{0}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0}{x_4}$. By Definition 3.8, it follows

that $\overline{U}_{\mathscr{C}} = \underline{U}_{\mathscr{C}} = \frac{0.3}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4} \neq U$. For $X = \frac{0.4}{x_1} + \frac{0}{x_2} + \frac{0.1}{x_3} + \frac{0.5}{x_4}$ and $Y = \frac{0}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3} + \frac{0}{x_4}$, we have that $\overline{X}_{\mathscr{C}} = \frac{0.3}{x_1} + \frac{0}{x_2} + \frac{0.5}{x_3} + \frac{0.4}{x_4}$, $\overline{Y}_{\mathscr{C}} = \frac{0}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0}{x_4}$ and $\overline{(X \cup Y)}_{\mathscr{C}} = \frac{0.3}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$. Consequently, $\overline{(X \cup Y)}_{\mathscr{C}} \neq \overline{X}_{\mathscr{C}} \cup \overline{Y}_{\mathscr{C}}$ and $X \nsubseteq \overline{X}_{\mathscr{C}}$.

Some relationships among $\underline{X}_{\mathscr{C}}$, $\overline{X}_{\mathscr{C}}$ and X are explored in the following.

Proposition 3.12 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, and $X \subseteq U$.*

(1) *If $\underline{X}_{\mathscr{C}} = X$, then $\overline{X}_{\mathscr{C}} = \underline{X}_{\mathscr{C}}$;*

(2) *If $\underline{X}_{\mathscr{C}} = X$, then $\overline{X}_{\mathscr{C}} = X$;*

(3) *$\underline{X}_{\mathscr{C}} = X$ if and only if X is a union of elements in $\mathscr{C}$;*

(4) *If X is a union of elements in $\mathscr{C}$, then $\overline{X}_{\mathscr{C}} = X$.*

Next, an example is given to illustrate that the converses of Proposition 3.12(1), (2) and (4) do not hold generally.

Example 3.13 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C} = \{C_{17}, C_{18}, C_{19}, C_{20}\}$, where $C_{17} = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4}$, $C_{18} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4}$, $C_{19} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0.1}{x_4}$, and $C_{20} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.2}{x_4}$. For $X = \frac{0.5}{x_1} + \frac{0.5}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$, it follows that $\underline{X}_{\mathscr{C}} = \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} = \overline{X}_{\mathscr{C}}$. But X is not a union of some subsets in the fuzzy covering $\mathscr{C}$. For $Y = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4}$, according to Definition 3.1, it follows that $C_{x_1} = C_{x_2} = C_{x_3} = C_{x_4} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4}$. Then we have that $\overline{Y} = Y$, but Y is not a union of some elements of $\mathscr{C}$.

From Proposition 3.10, we see that $\underline{X}_{\mathscr{C}} \cap \underline{Y}_{\mathscr{C}} = \underline{(X \cap Y)}_{\mathscr{C}}$ does not hold generally for any $X, Y \subseteq U$ in the fuzzy covering approximation space. But if $\underline{X}_{\mathscr{C}} \cap \underline{Y}_{\mathscr{C}} = \underline{(X \cap Y)}_{\mathscr{C}}$ for any $X, Y \subseteq U$, then we can obtain the following results.

Proposition 3.14 *If $\underline{X}_{\mathscr{C}} \cap \underline{Y}_{\mathscr{C}} = \underline{(X \cap Y)}_{\mathscr{C}}$ for any $X, Y \subseteq U$, then $C_1 \cap C_2 = \emptyset$ or $C_1 \cap C_2$ is a union of elements of $\mathscr{C}$ for any $C_1, C_2 \in \mathscr{C}$.*

Proof. Taking any $C_1, C_2 \in \mathscr{C}$, it follows that $\underline{C_1}_{\mathscr{C}} \cap \underline{C_2}_{\mathscr{C}} = \underline{(C_1 \cap C_2)}_{\mathscr{C}} = C_1 \cap C_2$. By Proposition 3.12, we have that $C_1 \cap C_2 = \emptyset$ or $C_1 \cap C_2$ is a union of elements of $\mathscr{C}$ for any $C_1, C_2 \in \mathscr{C}$. $\square$

This proposition shows that the intersection of two elementary elements in a fuzzy covering $\mathscr{C}$ can be represented as a union of elements of $\mathscr{C}$ if $\underline{X}_{\mathscr{C}} \cap \underline{Y}_{\mathscr{C}} = \underline{(X \cap Y)}_{\mathscr{C}}$ for any $X, Y \subseteq U$.

It is clear that $\underline{X}_{\mathscr{C}} \subseteq \underline{X}_{Cov(\mathscr{C})}$ and $\overline{X}_{\mathscr{C}} = \overline{X}_{Cov(\mathscr{C})}$ for any $X \subseteq U$ in the classical covering approximation space $(U, \mathscr{C})$. But they do not necessarily hold in the fuzzy covering approximation space. To illustrate this point, we employ the following example.

Example 3.15 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C} = \{C_{21}, C_{22}, C_{23}\}$, where $C_{21} = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0}{x_4}$, $C_{22} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0}{x_4}$ and $C_{23} = \frac{0.1}{x_1} + \frac{0}{x_2} + \frac{0.4}{x_3} + \frac{0.5}{x_4}$. According to Definition 3.1, we have that $C_{x_1} = \frac{0.1}{x_1} + \frac{0}{x_2} + \frac{0.2}{x_3} + \frac{0}{x_4}$, $C_{x_2} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0}{x_4}$, $C_{x_3} = \frac{0.1}{x_1} + \frac{0}{x_2} + \frac{0.2}{x_3} + \frac{0}{x_4}$ and $C_{x_4} = \frac{0.1}{x_1} + \frac{0}{x_2} + \frac{0.4}{x_3} + \frac{0.5}{x_4}$. Taking $X = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0}{x_4}$, according to Definition 3.8, it follows that $\underline{X}_{\mathscr{C}} = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0}{x_4}$ and $\underline{X}_{Cov(\mathscr{C})} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0}{x_4}$. Consequently, $\underline{X}_{\mathscr{C}} \nsubseteq \underline{X}_{Cov(\mathscr{C})}$. Similarly, we obtain that $\overline{X}_{\mathscr{C}} \neq \overline{X}_{Cov(\mathscr{C})}$.

It is well known that the upper approximation can be represented with neighborhoods in the classical covering approximation space. But we do not have the same result in the fuzzy covering approximation space.

Theorem 3.16 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space. Then $\bigcup\{C_x | X(x) > 0\} \subseteq \overline{X}_{\mathscr{C}}$ holds for $X \subseteq U$.*

Proof. Taking any $X \subseteq U$, according to Definition 3.8, we see that $\overline{X}_{\mathscr{C}} = (\bigcup\{C_x | X(x) > 0 \text{ and } \underline{X}_{\mathscr{C}}(x) = 0\}) \cup \underline{X}_{\mathscr{C}}$ and $\bigcup\{C_x | X(x) > 0\} = (\bigcup\{C_x | X(x) > 0 \text{ and } \underline{X}_{\mathscr{C}}(x) = 0\}) \cup (\bigcup\{C_x | \underline{X}_{\mathscr{C}}(x) > 0\})$. It follows that there exist $C \in \mathscr{C}$ and $C \subseteq X$ for any x satisfying $\underline{X}(x) > 0$. Consequently, $C_x \subseteq C \subseteq X$. Hence, $\bigcup\{C_x | \underline{X}_{\mathscr{C}}(x) > 0\} \subseteq X$. Therefore, $\bigcup\{C_x | X(x) > 0\} \subseteq \overline{X}_{\mathscr{C}}$ holds for $X \subseteq U$. □

We see that $\overline{X}_{\mathscr{C}} \subseteq \bigcup\{C_x | X(x) > 0\}$ does not necessarily hold for any $X \subseteq U$. So the upper approximation can not be represented with neighborhoods only in the fuzzy covering approximation space. To show this point, we give an example below.

Example 3.17 Let $U = \{x_1, x_2, x_3, x_4\}$, and $\mathscr{C} = \{C_{17}, C_{18}, C_{19}, C_{20}\}$. For $X = \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4}$, it follows that $\underline{X}_{\mathscr{C}} = \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} = \overline{X}_{\mathscr{C}}$. Furthermore, $\bigcup\{C_x | X(x) > 0\} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4}$. Obviously, $\overline{X}_{\mathscr{C}} \nsubseteq \bigcup\{C_x | X(x) > 0\}$.

We now investigate the relationship between the lower and upper approximation operators.

Theorem 3.18 *Let U be a non-empty universe of discourse, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$. If $\underline{X}_{\mathscr{C}_1} = \underline{X}_{\mathscr{C}_2}$ for any $X \subseteq U$, then $\overline{X}_{\mathscr{C}_1} = \overline{X}_{\mathscr{C}_2}$.*

Proof. By Definition 3.1, we have that $C_{1x} = \bigcap\{C_i|C_i(x) > 0, C_i \in \mathscr{C}_1, i \in I\}$ and $C_{2x} = \bigcap\{C_j|C_j(x) > 0, C_j \in \mathscr{C}_2, j \in J\}$. For any C_i, where $i \in I$, $C_i = \underline{C_i}_{\mathscr{C}_1} = \underline{C_i}_{\mathscr{C}_2}$. So there exists at least $C_j \in \mathscr{C}_2$ such that $C_j \subseteq C_i$ and $C_j(x) > 0$. Hence, $C_{2x} \subseteq C_{1x}$. Similarly, we obtain that $C_{1x} \subseteq C_{2x}$. Therefore, $\overline{X}_{\mathscr{C}_1} = \overline{X}_{\mathscr{C}_2}$. □

From Theorem 3.18, we see that the lower and upper approximation operations are not independent in the fuzzy covering approximation space. Concretely, the lower approximation operation dominates the upper one.

Theorem 3.19 *Let U be a non-empty universe of discourse, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$. Then $\underline{C}_{\mathscr{C}_1} = \underline{C}_{\mathscr{C}_2}$ holds for any $C \in \mathscr{C}_1 \cup \mathscr{C}_2$ if and only if $\underline{X}_{\mathscr{C}_1} = \underline{X}_{\mathscr{C}_2}$ for any $X \subseteq U$.*

Proof. Taking any $X \subseteq U$, by Definition 3.8, it follows that $\underline{X}_{\mathscr{C}_1} = \bigcup\{C_i|C_i \subseteq X, C_i \in \mathscr{C}_1, i \in I\}$. For any $C_i \subseteq \underline{X}_{\mathscr{C}_1}$, we have that $C_i = \underline{C_i}_{\mathscr{C}_1} = \underline{C_i}_{\mathscr{C}_2} = \bigcup\{C_{ij}|C_{ij} \in \mathscr{C}_2, C_{ij} \subseteq X, i \in I, j \in J\}$. It implies that $\underline{X}_{\mathscr{C}_1} \subseteq \underline{X}_{\mathscr{C}_2}$. Analogously, it follows that $\underline{X}_{\mathscr{C}_2} \subseteq \underline{X}_{\mathscr{C}_1}$. Thereby, $\underline{X}_{\mathscr{C}_1} = \underline{X}_{\mathscr{C}_2}$ for any $X \subseteq U$.

The converse is obvious by Definition 3.8. □

This result indicates that each elementary set in a fuzzy covering is definable in the other fuzzy covering if and only if two fuzzy coverings of a universe give the same lower approximations.

3.2 The fuzzy subcovering and its properties

It is well-known that the classical upper approximation based on neighborhoods can be defined equivalently by using a family of subcoverings. In this subsection, we propose the notion of fuzzy subcoverings and investigate the relationship between the upper approximation based on neighborhoods and subcoverings in the fuzzy covering approximation space.

Definition 3.20 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, $X \subseteq U$, and $\mathscr{C}' \subseteq \mathscr{C}$. If $X \subseteq \bigcup\{C|C \in \mathscr{C}'\}$, then $\mathscr{C}'$ is called a fuzzy subcovering of X.*

In other words, the fuzzy subcovering of X is a subset of $\mathscr{C}$ which covers X. Obviously, $\mathscr{C}$ is the maximum fuzzy subcovering for $X \subseteq U$ if $X \subseteq \bigcup \mathscr{C}$. In this work, we denote the set of all the fuzzy subcoverings of X as $FC(X)$.

Theorem 3.21 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space. Then $\overline{X}_{\mathscr{C}} \subseteq \bigcap\{\bigcup\{C|C \in \mathscr{C}'\}|\mathscr{C}' \in FC(X)\}$ holds for any $X \subseteq U$.*

Proof. Taking any $X \subseteq U$, by Definition 3.8, it follows that $\overline{X}_{\mathscr{C}} = (\bigcup\{C_x | X(x) > 0 \text{ and } \underline{X}_{\mathscr{C}}(x) = 0\}) \cup \underline{X}_{\mathscr{C}}$. By Proposition 3.9, it implies that $\underline{X} \subseteq X \subseteq \bigcup\{C | C \in \mathscr{C}' \in FC(X)\}$. Evidently, $C_x \subseteq \bigcup\{C | C \in \mathscr{C}' \in FC(X)\}$ for any $x \in U$ satisfying $X(x) > 0$. Thereby, $\overline{X}_{\mathscr{C}} \subseteq \bigcap\{\bigcup\{C | C \in \mathscr{C}'\} | \mathscr{C}' \in FC(X)\}$ holds for any $X \subseteq U$. □

However, $\bigcap\{\bigcup\{C | C \in \mathscr{C}'\} | \mathscr{C}' \in FC(X)\} \subseteq \overline{X}_{\mathscr{C}}$ does not hold generally. That is, the upper approximation may not be represented with a family of fuzzy subcoverings of X as the classical covering approximation space, which is shown by the following example.

Example 3.22 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C} = \{C_{24}, C_{25}, C_{26}\}$, where $C_{24} = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0.1}{x_4}$, $C_{25} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.1}{x_3} + \frac{0.2}{x_4}$ and $C_{26} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0.1}{x_4}$. According to Definition 3.20, we obtain all fuzzy subcoverings of $X = \frac{0.1}{x_1} + \frac{0}{x_2} + \frac{0.2}{x_3} + \frac{0}{x_4}$ as $FC(X) = \{\{C_{24}\}, \{C_{26}\}, \{C_{24}, C_{25}\}, \{C_{24}, C_{26}\}, \{C_{25}, C_{26}\}, \{C_{24}, C_{25}, C_{26}\}\}$. It follows that $\bigcap\{\bigcup\{C | C \in \mathscr{C}'\} | \mathscr{C}' \in FC(X)\} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0.1}{x_4}$, but $\overline{X}_{\mathscr{C}} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4}$. Therefore, $\bigcap\{\bigcup\{C | C \in \mathscr{C}'\} | \mathscr{C}' \in FC(X)\} \nsubseteq \overline{X}_{\mathscr{C}}$.

According to Theorems 3.16 and 3.21, we have that $\bigcup\{C_x | X(x) > 0\} \subseteq \overline{X}_{\mathscr{C}} \subseteq \bigcap\{\bigcup\{C | C \in \mathscr{C}'\} | \mathscr{C}' \in FC(X)\}$ for any $X \subseteq U$.

Sometimes, the fuzzy covering $\mathscr{C}$ of U is a trivial subcovering of $X \subseteq U$. Specially, we do not take $\mathscr{C}$ into account in the following situation.

Proposition 3.23 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, and $X \subseteq U$. If $|FC(X)| \geq 2$, where $|FC(X)|$ stands for the cardinality of $FC(X)$, then $\bigcap\{\bigcup\{C | C \in \mathscr{C}'\} | \mathscr{C}' \in FC(X)\} = \bigcap\{\bigcup\{C | C \in \mathscr{C}'\} | \mathscr{C}' \in FC(X) - \{\mathscr{C}\}\}$.*

Proof. Straightforward. □

3.3 The irreducible and reducible elements of a fuzzy covering

In this subsection, we provide the concepts of reducible and irreducible elements to formally investigate the relationship among elementary elements of a fuzzy covering. Although several theorems in this subsection are special cases of [33], they don't give their proofs. To better understand the following results, we prove them concretely in the following.

Definition 3.24 *[33] Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, and $C \in \mathscr{C}$. If C can not be written as a union of some sets in $\mathscr{C} - \{C\}$, then C is called an irreducible element. Otherwise, C is called*

a reducible element.

It is obvious that the concept of the irreducible element in a fuzzy covering approximation space is an extension of the notion of the irreducible element in a covering approximation space, and the irreducible element can be used for the definition of reducts of fuzzy coverings.

Proposition 3.25 *[33] Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, and C a reducible element of $\mathscr{C}$. Then $\mathscr{C} - \{C\}$ is still a fuzzy covering of U.*

In other words, a fuzzy covering of a universe deleting all reducible elements is a fuzzy covering, and the rest elements are irreducible.

Definition 3.26 *[33] Let $(U, \mathscr{C})$ be a fuzzy covering approximation space. If every element of $\mathscr{C}$ is an irreducible element, then $\mathscr{C}$ is irreducible. Otherwise, $\mathscr{C}$ is reducible.*

Next, we discuss the properties of reducible elements of a fuzzy covering.

Theorem 3.27 *[33] Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, C a reducible element of $\mathscr{C}$, and $C_0 \in \mathscr{C} - \{C\}$. Then C_0 is a reducible element of $\mathscr{C}$ if and only if it is a reducible element of $\mathscr{C} - \{C\}$.*

Proof. We assume that C_0 is a reducible element of $\mathscr{C}$. It follows that we can express C_0 as a union of subset of $\mathscr{C} - \{C_0\}$, denoted as $C_1, C_2, ..., C_N$. If there exists no set which is equal to C in $\{C_1, C_2, ..., C_N\}$, then C_0 is a reducible element of $\mathscr{C} - \{C\}$. If there is a set which is equal to C in $\{C_1, C_2, ..., C_N\}$, taking $C_1 = C$, then C_1 is the union of some sets $\{D_1, D_2, ..., D_M\}$ in $\mathscr{C} - \{C\}$. Consequently, we obtain that $C_0 = D_1 \cup D_2 \cup ...D_M \cup C_2 \cup ... \cup C_N$. Clearly, $D_1, D_2, ..., D_M, C_2, ..., C_N$ are not equal to either C_0 or C. So C is a reducible element of $\mathscr{C} - \{C\}$.

Since C_0 is a reducible element of $\mathscr{C} - \{C\}$, it can be expressed as a union of some sets in $\mathscr{C} - \{C, C_0\}$. We can express it as a union of some sets in $\mathscr{C} - \{C_0\}$. Therefore, C_0 is a reducible element of $\mathscr{C}$. □

Next, we investigate the relationship between the approximation operations and the reducible elements in the fuzzy covering approximation space.

Theorem 3.28 *[33] Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, and C a reducible element of $\mathscr{C}$. Then $\underline{X}_{\mathscr{C}} = \underline{X}_{\mathscr{C}-\{C\}}$ holds for any $X \subseteq U$.*

Proof. Taking any $X \subseteq U$, by Definition 3.24, it follows that $\underline{X}_{\mathscr{C}-\{C\}} \subseteq \underline{X}_{\mathscr{C}} \subseteq X$. Moreover, there exist $C_1, C_2, ..., C_N$ such that $\underline{X}_{\mathscr{C}} = C_1 \cup C_2 \cup ... \cup C_N$. If none of $C_1, C_2, ..., C_N$ is equal to C, then they belong to $\mathscr{C} - \{C\}$. Consequently, $C_1, C_2, ..., C_N$ are all the subsets of $\underline{X}_{\mathscr{C}-\{C\}}$. If there is a set which is equal to C, then we take $C = C_1$. Since C is a reducible element of $\mathscr{C}$, C can be expressed as some sets in $\mathscr{C} - \{C\}$ such that $C = D_1 \cup D_2 \cup ... \cup D_M$. Hence, $\underline{X}_{\mathscr{C}} = D_1 \cup D_2 \cup ... \cup D_M \cup C_2 \cup ... \cup C_N$. It implies that $\underline{X}_{\mathscr{C}} \subseteq \underline{X}_{\mathscr{C}-\{C\}}$. Therefore, $\underline{X}_{\mathscr{C}} = \underline{X}_{\mathscr{C}-\{C\}}$ holds for any $X \subseteq U$. □

In other words, the lower approximation of any $X \subseteq U_1$ in $\mathscr{C}$ is the same as that in $\mathscr{C} - \{C\}$ if C is reducible.

Corollary 3.29 *[33] Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, and C a reducible element of $\mathscr{C}$. Then $\overline{X}_{\mathscr{C}} = \overline{X}_{\mathscr{C}-\{C\}}$ holds for any $X \subseteq U$.*

Proof. Straightforward from Theorems 3.27 and 3.28. □

In this sequel, we use $RED(\mathscr{C})$ to represent the set of all irreducible elements of a fuzzy covering $\mathscr{C}$. It is easy to see that $RED(\mathscr{C}) = RED(RED(\mathscr{C}))$. Next, we study the relationship between $RED(\mathscr{C})$ and the lower and upper approximation operations.

Corollary 3.30 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space. Then $\underline{X}_{\mathscr{C}} = \underline{X}_{RED(\mathscr{C})}$ holds for any $X \subseteq U$.*

Proof. Straightforward from Theorem 3.28. □

Corollary 3.31 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space. Then $\overline{X}_{\mathscr{C}} = \overline{X}_{RED(\mathscr{C})}$ holds for any $X \subseteq U$.*

Proof. Straightforward from Corollary 3.29. □

Based on Theorem 3.28, Corollaries 3.29, 3.30 and 3.31, we obtain the following theorem.

Theorem 3.32 *Let U be a universe, $\mathscr{C}_1$ and $\mathscr{C}_2$ two irreducible fuzzy coverings of U. If $\underline{X}_{\mathscr{C}_1} = \underline{X}_{\mathscr{C}_2}$ for any $X \subseteq U$. then the two fuzzy coverings are the same.*

Proof. Taking any $C \in \mathscr{C}_1$, by Definition 3.8, it follows that $\underline{C}_{\mathscr{C}_1} = C = \underline{C}_{\mathscr{C}_2}$. Consequently, C is the union of some sets of $\mathscr{C}_2$ such that $C = C_1 \cup C_2 \cup ... \cup C_N$. Similarly, there exist $D_{i1}, D_{i2}, ..., D_{iM(i)} \in \mathscr{C}_1$ such that $C_i = D_{i1} \cup D_{i2} \cup ... \cup D_{iM(i)}$. Hence, $C = D_{11} \cup D_{12} \cup ... \cup D_{N1} \cup D_{N2} \cup ... \cup D_{NM(N)}$. Since C is

irreducible, $C = D_{ij}$ for all i, j. It implies that C is an element of $\mathscr{C}_2$. On the other hand, any element of $\mathscr{C}_2$ is an element of $\mathscr{C}_1$. Therefore, the two fuzzy coverings $\mathscr{C}_1$ and $\mathscr{C}_2$ are the same. □

Corollary 3.33 *Let U be a universe, $\mathscr{C}_1$ and $\mathscr{C}_2$ two irreducible fuzzy coverings of U. If $\overline{X}_{\mathscr{C}_1} = \overline{X}_{\mathscr{C}_2}$ for any $X \subseteq U$, then the two fuzzy coverings are the same.*

Proof. The proof is similar to that in Theorem 3.32. □

Theorem 3.34 *Let U be a non-empty universe of discourse, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$. Then $\underline{X}_{\mathscr{C}_1} = \underline{X}_{\mathscr{C}_2}$ holds for any $X \subseteq U$ if and only if $RED(\mathscr{C}_1) = RED(\mathscr{C}_2)$.*

Proof. Since $\underline{X}_{\mathscr{C}_1} = \underline{X}_{\mathscr{C}_2}$ for any $X \subseteq U$, $\underline{C}_{\mathscr{C}_1} = \underline{C}_{\mathscr{C}_2}$ for any $C \in \mathscr{C}_1 \cup \mathscr{C}_2$. Taking any $C \in RED(\mathscr{C}_1)$, it follows that $C = \bigcup\{C_i | C_i \in RED(\mathscr{C}_2), i \in I\} = \bigcup\{\bigcup\{C_{ij} | C_{ij} \in \mathscr{C}_1, i \in I\} | j \in J\}$. It implies that $C \in \mathscr{C}_2$. Hence, $RED(\mathscr{C}_1) \subseteq RED(\mathscr{C}_2)$. Similarly, we obtain that $RED(\mathscr{C}_2) \subseteq RED(\mathscr{C}_1)$. Therefore, $RED(\mathscr{C}_1) = RED(\mathscr{C}_2)$.

The converse is obvious by Definitions 3.8 and 3.24. □

It can be seen from Theorem 3.34 that two fuzzy coverings of a universe generate the same lower approximation if and only if there exist the same irreducible elements in these fuzzy coverings.

To illustrate Theorem 3.34, we supply the following example.

Example 3.35 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C}_1 = \{C_{17}, C_{18}, C_{19}, C_{20}, C_{27}, C_{28}\}$, $\mathscr{C}_2 = \{C_{17}, C_{18}, C_{19}, C_{20}, C_{29}, C_{30}\}$, where $C_{27} = \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4}$, $C_{28} = \frac{0.2}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0.1}{x_4}$, $C_{29} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4}$ and $C_{30} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0.1}{x_3} + \frac{0.2}{x_4}$. Obviously, we obtain that $RED(\mathscr{C}_1) = RED(\mathscr{C}_2) = \{C_{17}, C_{18}, C_{19}, C_{20}\}$.

Corollary 3.36 *Let U be a non-empty universe of discourse, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$. If $\overline{X}_{\mathscr{C}_1} = \overline{X}_{\mathscr{C}_2}$ for any $X \subseteq U$ if and only if $RED(\mathscr{C}_1) = RED(\mathscr{C}_2)$.*

From Corollary 3.36, we see that they have the same irreducible elements if and only if two fuzzy coverings of a universe generate the same upper approximation.

Corollary 3.37 *Let $(U, \mathscr{C}_1)$ be a fuzzy covering approximation space, $\mathscr{C}_2 = \{\bigcup_{C \in \mathscr{C}'} C | \emptyset \neq \mathscr{C}' \subseteq \mathscr{C}_1\}$, and $X \subseteq U$. Then $\underline{X}_{\mathscr{C}_1} = \underline{X}_{\mathscr{C}_2}$ and $\overline{X}_{\mathscr{C}_1} = \overline{X}_{\mathscr{C}_2}$.*

Proof. By Definition 3.24, we observe that $RED(\mathscr{C}_1) = RED(\mathscr{C}_2)$. Therefore, $\underline{X}_{\mathscr{C}_1} = \underline{X}_{\mathscr{C}_2}$ and $\overline{X}_{\mathscr{C}_1} = \overline{X}_{\mathscr{C}_2}$. □

We also investigate the relationship between the reducible elements and the neighborhood operator in the fuzzy covering approximation space.

Theorem 3.38 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, and C a reducible element of $\mathscr{C}$. Then C_x in $\mathscr{C} - \{C\}$ is the same as that in $\mathscr{C}$ for any $x \in U$.*

Proof. By Definitions 3.1 and 3.24, we have that $C_x = \bigcap\{C_i | C_i(x) > 0, C_i \in \mathscr{C}\} = \bigcap\{C_i | C_i(x) > 0, C_i \in \mathscr{C} - \{C\}\}$ for any $x \in U$. Therefore, C_x in $\mathscr{C} - \{C\}$ is the same as that in $\mathscr{C}$ for any $x \in U$. □

That is to say, if we delete some reducible elements in the fuzzy covering, then it will not change the neighborhood C_x for any $x \in U$.

Corollary 3.39 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space. Then C_x in $RED(\mathscr{C})$ is the same as that in $\mathscr{C}$ for any $x \in U$.*

Proof. Straightforward from Theorem 3.38. □

Corollary 3.39 indicates that $RED(\mathscr{C})$ and $\mathscr{C}$ generate the same neighborhood C_x for any $x \in U$ in the fuzzy covering approximation space.

Corollary 3.40 *Let U be a non-empty universe of discourse, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$. If $RED(\mathscr{C}_1) = RED(\mathscr{C}_2)$, then C_x in $\mathscr{C}_1$ is the same as that in $\mathscr{C}_2$ for any $x \in U$.*

Proof. Straightforward from Corollary 3.39. □

By Corollary 3.40, if there exist the same irreducible elements in two fuzzy coverings $\mathscr{C}_1$ and $\mathscr{C}_2$ of U, then they generate the same neighborhood C_x for any $x \in U$.

Theorem 3.41 *Let U be a non-empty universe of discourse, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$. If $\overline{C}_{\mathscr{C}_1} = \overline{C}_{\mathscr{C}_2}$ for any $C \in \mathscr{C}_1 \cup \mathscr{C}_2$, then $\bigcup\{C_{1x} | X(x) > 0\} = \bigcup\{C_{2x} | X(x) > 0\}$ for any $X \subseteq U$.*

Proof. By Definition 3.1, we have that $C_{1x} = \bigcap\{C_i | C_i(x) > 0, C_i \in \mathscr{C}_1, i \in I\}$ and $C_{2x} = \bigcap\{C_j | C_j(x) > 0, C_j \in \mathscr{C}_2, j \in J\}$ for any $x \in U$. Assume that there exists $x \in U$ such that $X(x) > 0$ and $C_{1x} \neq C_{2x}$. Without loss of generality, there is $y \in U$ such that $(C_{1x})(y) > 0$ and $(C_{2x})(y) = 0$. Obviously, $y \neq x$. Hence, there exist $C_j(y) = 0$ and $C_j(x) > 0$. But $C_j = \overline{C}_{j\mathscr{C}_2} = \overline{C}_{j\mathscr{C}_1} \supseteq \bigcup\{C_{1z} | C_j(z) > 0\} \supseteq C_{1x}$. It implies that $C_j(y) > 0$, which is a contradiction. Consequently, $C_{1x} = C_{2x}$ for any $x \in U$. Therefore, $\bigcup\{C_{1x} | X(x) > 0\} = \bigcup\{C_{2x} | X(x) > 0\}$ for any $X \subseteq U$. □

Theorem 3.41 shows that two fuzzy coverings of a universe generate the same neighborhood C_x for any $x \in U$ if each elementary element has the same lower approximation in two fuzzy coverings.

3.4 The non-intersectional and intersectional elements of fuzzy coverings, the union and intersection operations on fuzzy coverings

For any universal set U, we denote $CC(U)$ as the set of all coverings of U. It is well-known that the number of possible coverings for a set U of n elements is

$$|CC(U)| = \frac{1}{2}\sum_{k=0}^{n}(\frac{n}{k})2^{2^{n-k}},$$

the first few of which are 1, 5, 109, 32297, 2147321017. Since $C(U)$ contains a larger number of fuzzy coverings than $CC(U)$ in practice, it is of interest to investigate the relationship between fuzzy coverings. In this subsection, we introduce several operations on fuzzy coverings and study their basic properties for facilitating the computation of fuzzy coverings.

Definition 3.42 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, and $C \in \mathscr{C}$. If C can not be written as an intersection of some sets in $\mathscr{C} - \{C\}$, then C is called a non-intersectional element. Otherwise, C is called an intersectional element.*

For simplicity, we use $IS(\mathscr{C})$ to represent the set of all non-intersectional elements of $\mathscr{C}$. It is easy to see that $IS(\mathscr{C}) = IS(IS(\mathscr{C}))$. Notice that the function $IS : C(U) \longrightarrow C(U)$ that maps $\mathscr{C}$ to $IS(\mathscr{C})$ is well-defined. Hence, we may view IS as a unary operator on $C(U)$.

We employ an example to illustrate the non-intersectional and intersectional elements in the following.

Example 3.43 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C} = \{C_{17}, C_{18}, C_{19}, C_{20}, C_{31}\}$, where $C_{31} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4}$. By Definition 3.42, we have that $IS(\mathscr{C}) = \{C_{17}, C_{18}, C_{19}, C_{20}\}$.

Proposition 3.44 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space. Then $\bigcup\{C_x|X(x) > 0\} = \bigcup\{C_{IS(\mathscr{C})x}| X(x) > 0\}$ and $\underline{X}_{IS(\mathscr{C})} \subseteq \underline{X}_{\mathscr{C}}$ for any $X \subseteq U$.*

Proof. By Definition 3.42, it follows that $C_x = C_{IS(\mathscr{C})x}$ for any $x \in U$. Consequently, $\bigcup\{C_x|X(x) > 0\} = \bigcup\{C_{IS(\mathscr{C})x}|X(x) > 0\}$ for any $X \subseteq U$. Furthermore, since $IS(\mathscr{C}) \subseteq \mathscr{C}$, we have that $\underline{X}_{IS(\mathscr{C})} \subseteq \underline{X}_{\mathscr{C}}$ for any $X \subseteq U$. $\square$

We observe that the neighborhood C_x generated in the fuzzy covering $\mathscr{C}$ is the same as that generated in all non-intersectional elements of $\mathscr{C}$. On the other hand, $\overline{X}_{\mathscr{C}} \subseteq \overline{X}_{IS(\mathscr{C})}$ does not necessarily hold for any $X \subseteq U$. An example is given to illustrate this point.

Example 3.45 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C} = \{C_{32}, C_{33}, C_{34}, C_{35}, C_{36}, C_{37}, C_{38}, C_{39}\}$, where $C_{32} = \frac{0.1}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0}{x_4}$, $C_{33} = \frac{0}{x_1} + \frac{0.1}{x_2} + \frac{0}{x_3} + \frac{0}{x_4}$, $C_{34} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0}{x_3} + \frac{0}{x_4}$, $C_{35} = \frac{0.1}{x_1} + \frac{0}{x_2} + \frac{0.1}{x_3} + \frac{0}{x_4}$, $C_{36} = \frac{0.4}{x_1} + \frac{0.2}{x_2} + \frac{0.1}{x_3} + \frac{0}{x_4}$, $C_{37} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0}{x_3} + \frac{0.1}{x_4}$, $C_{38} = \frac{0.1}{x_1} + \frac{0}{x_2} + \frac{0.1}{x_3} + \frac{0.5}{x_4}$ and $C_{39} = \frac{0}{x_1} + \frac{0.1}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4}$. Evidently, $IS(\mathscr{C}) = \{C_{36}, C_{37}, C_{38}, C_{39}\}$. Taking $X = \frac{0.4}{x_1} + \frac{0.2}{x_2} + \frac{0}{x_3} + \frac{0}{x_4}$, according to Definitions 3.8 and 3.42, we obtain that $\overline{X}_{\mathscr{C}} = \frac{0.1}{x_1} + \frac{0.2}{x_2} + \frac{0}{x_3} + \frac{0}{x_4}$ and $\overline{X}_{IS(\mathscr{C})} = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0}{x_3} + \frac{0}{x_4}$. Thereby, $\overline{X}_{\mathscr{C}} \nsubseteq \overline{X}_{IS(\mathscr{C})}$.

Following, we present a theorem for the intersection element of a fuzzy covering.

Theorem 3.46 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, C an intersection element of $\mathscr{C}$, and $C_0 \in \mathscr{C} - \{C\}$. Then C_0 is an intersection element of $\mathscr{C}$ if and only if it is an intersection element of $\mathscr{C} - \{C\}$.*

Proof. The proof is similar to that in Theorem 3.27. □

Next, we present the notions of the union and intersection operations on fuzzy coverings, and investigate their basic properties.

Definition 3.47 *Let U be a non-empty universe of discourse, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$. If*

$$\mathscr{C}_1 \cup \mathscr{C}_2 = \{C | C \in \mathscr{C}_1 \text{ or } C \in \mathscr{C}_2\},$$

then $\mathscr{C}_1 \cup \mathscr{C}_2$ is called the union of $\mathscr{C}_1$ and $\mathscr{C}_2$.

It is obvious that the union operation is to collect all elementary elements in each fuzzy covering.

Proposition 3.48 *Let U be a non-empty universe of discourse, $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$, and $X \subseteq U$. Then $\underline{X}_{\mathscr{C}_i} \subseteq \underline{X}_{\mathscr{C}_1 \cup \mathscr{C}_2}$, where $i = 1, 2$.*

Proof. According to Definition 3.8, we have that $\underline{X}_{\mathscr{C}_i} = \bigcup\{C \in \mathscr{C}_i | C \subseteq X\} \subseteq (\bigcup\{C \in \mathscr{C}_1 | C \subseteq X\}) \bigcup (\bigcup\{C \in \mathscr{C}_2 | C \subseteq X\}) = \underline{X}_{\mathscr{C}_1 \cup \mathscr{C}_2}$. Thereby, $\underline{X}_{\mathscr{C}_i} \subseteq \underline{X}_{\mathscr{C}_1 \cup \mathscr{C}_2}$, where $i = 1, 2$. □

The following example shows that the converse of Proposition 3.48 does not hold generally.

Example 3.49 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C}_1 = \{C_{21}, C_{22}, C_{23}\}$, and $\mathscr{C}_2 = \{C_{1x_1}, C_{1x_2}, C_{1x_3}, C_{1x_4}\}$. Taking $X = \frac{0.2}{x_1}+\frac{0.5}{x_2}+\frac{0.6}{x_3}+\frac{0.1}{x_4}$. By Definition 3.8, we have that $\overline{X}_{\mathscr{C}_1} = \frac{0.2}{x_1}+\frac{0.4}{x_2}+\frac{0.5}{x_3}+\frac{0.5}{x_4}$, $\overline{X}_{\mathscr{C}_2} = \frac{0.1}{x_1}+\frac{0.1}{x_2}+\frac{0.4}{x_3}+\frac{0.5}{x_4}$, $\underline{X}_{\mathscr{C}_1 \bigcup \mathscr{C}_2} = \frac{0.2}{x_1}+\frac{0.4}{x_2}+\frac{0.5}{x_3}+\frac{0}{x_4}$ and $\overline{X}_{\mathscr{C}_1 \bigcup \mathscr{C}_2} = \frac{0.2}{x_1}+\frac{0.4}{x_2}+\frac{0.5}{x_3}+\frac{0.5}{x_4}$. Obviously, $\overline{X}_{\mathscr{C}_1 \bigcup \mathscr{C}_2} \nsubseteq \overline{X}_{\mathscr{C}_2}$.

Definition 3.50 *Let U be a non-empty universe of discourse, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$. If*

$$\mathscr{C}_1 \cap \mathscr{C}_2 = \{C_{1x} \cap C_{2x} | C_{ix} \in Cov(\mathscr{C}_i), x \in U_1, i = 1, 2\},$$

then $\mathscr{C}_1 \cap \mathscr{C}_2$ is called the intersection of $\mathscr{C}_1$ and $\mathscr{C}_2$.

It is obvious that $C_{1x} = \bigcap\{C | C(x) > 0, C \in \mathscr{C}_1\}$ and $C_{2x} = \bigcap\{C' | C'(x) > 0, C' \in \mathscr{C}_2\}$ for any $x \in U_1$. So $\mathscr{C}_1 \bigcap \mathscr{C}_2$ is a fuzzy covering of U_1. Furthermore, if we take the value of membership degree from the set $\{0, 1\}$, then Definition 3.50 is the same as that in Definition 4.2 in [24].

It can be found that $\overline{X}_{\mathscr{C}_i} \subseteq \overline{X}_{\mathscr{C}_1 \bigcap \mathscr{C}_2}$ does not necessarily hold for any $X \subseteq U$ in the fuzzy covering approximation space, where $i = 1, 2$. To illustrate this point, we give the following example.

Example 3.51 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C}_1 = \{C_{21}, C_{22}, C_{23}\}$, and $\mathscr{C}_2 = \{\frac{0.2}{x_1}+\frac{0.1}{x_2}+\frac{0.4}{x_3}+\frac{0.5}{x_4}\}$. According to Definition 3.1, we have that $C_{1x_1} = \frac{0.1}{x_1}+\frac{0}{x_2}+\frac{0.2}{x_3}+\frac{0}{x_4}$, $C_{1x_2} = \frac{0.1}{x_1}+\frac{0.1}{x_2}+\frac{0.2}{x_3}+\frac{0}{x_4}$, $C_{1x_3} = \frac{0.1}{x_1}+\frac{0}{x_2}+\frac{0.2}{x_3}+\frac{0}{x_4}$ and $C_{1x_4} = \frac{0.1}{x_1}+\frac{0}{x_2}+\frac{0.4}{x_3}+\frac{0.5}{x_4}$. Taking $X = \frac{0.2}{x_1}+\frac{0.4}{x_2}+\frac{0.5}{x_3}+\frac{0.5}{x_4}$, it implies that $\underline{X}_{\mathscr{C}_1 \bigcap \mathscr{C}_2} = \frac{0.1}{x_1}+\frac{0.1}{x_2}+\frac{0.4}{x_3}+\frac{0.5}{x_4}$, $\underline{X}_{\mathscr{C}_1} = \frac{0.2}{x_1}+\frac{0.4}{x_2}+\frac{0.4}{x_3}+\frac{0.5}{x_4}$ and $\underline{X}_{\mathscr{C}_2} = \frac{0.2}{x_1}+\frac{0.1}{x_2}+\frac{0.4}{x_3}+\frac{0.5}{x_4}$. Clearly, $\overline{X}_{\mathscr{C}_1} \nsubseteq \overline{X}_{\mathscr{C}_1 \bigcap \mathscr{C}_2}$ and $\overline{X}_{\mathscr{C}_2} \nsubseteq \overline{X}_{\mathscr{C}_1 \bigcap \mathscr{C}_2}$.

By Definitions 3.4 and 3.50, we present the following proposition.

Proposition 3.52 *Let U be a non-empty universe of discourse, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$. Then $\mathscr{C}_1 \bigcap \mathscr{C}_2 = Cov(\mathscr{C}_1 \bigcup \mathscr{C}_2)$.*

Definition 3.53 *Let U be a non-empty universe of discourse, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$. If there exists $C^* \in \mathscr{C}_2$ such that $C \subseteq C^*$ for any $C \in \mathscr{C}_1$, then $\mathscr{C}_2$ is said to be coarser than $\mathscr{C}_1$, denoted as $\mathscr{C}_1 \leq \mathscr{C}_2$.*

In other words, there exists $C^* \in \mathscr{C}_2$ such that $C^*(x) \leq C(x)$ for each $C \in \mathscr{C}_1$ and $x \in U$ if $\mathscr{C}_1 \leq \mathscr{C}_2$.

Example 3.54 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C}_1 = \{C_{21}, C_{22}, C_{23}\}$, and $\mathscr{C}_2 = \{C_{40}, C_{41}\}$, where $C_{40} = \frac{0.2}{x_1}+\frac{0.4}{x_2}+\frac{0.5}{x_3}+\frac{0}{x_4}$ and $C_2 = \frac{0.3}{x_1}+\frac{0}{x_2}+\frac{0.6}{x_3}+\frac{0.5}{x_4}$. It is obvious that $\mathscr{C}_2$ is coarser than $\mathscr{C}_1$.

Proposition 3.55 *Let U be a non-empty universe of discourse, and $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3 \in C(U)$. Then*

(1) $\mathscr{C}_1 \cup \mathscr{C}_1 = \mathscr{C}_1$;

(2) $\mathscr{C}_1 \cap \mathscr{C}_1 \leq \mathscr{C}_1$;

(3) $\mathscr{C}_1 \cup \mathscr{C}_2 = \mathscr{C}_2 \cup \mathscr{C}_1$;

(4) $\mathscr{C}_1 \cap \mathscr{C}_2 = \mathscr{C}_2 \cap \mathscr{C}_1$;

(5) $(\mathscr{C}_1 \cup \mathscr{C}_2) \cup \mathscr{C}_3 = \mathscr{C}_1 \cup (\mathscr{C}_2 \cup \mathscr{C}_3)$;

(6) $(\mathscr{C}_1 \cap \mathscr{C}_2) \cap \mathscr{C}_3 = \mathscr{C}_1 \cap (\mathscr{C}_2 \cap \mathscr{C}_3)$;

(7) $\mathscr{C}_1 \cup (\mathscr{C}_1 \cap \mathscr{C}_2) \leq \mathscr{C}_1$;

(8) $\mathscr{C}_1 \cap (\mathscr{C}_1 \cup \mathscr{C}_2) \leq \mathscr{C}_1$.

Proof. Straightforward from Definitions 3.47, 3.50 and 3.53. □

Proposition 3.56 *Let U be a non-empty universe of discourse, and $C(U)$ the set of all fuzzy coverings of U. Then $(C(U), \cap, \cup)$ is a lattice.*

Proof. Given any $\mathscr{C}_1, \mathscr{C}_2 \in C(U)$, it is obvious that $\mathscr{C}_1 \cup \mathscr{C}_2 \in C(U)$ and $\mathscr{C}_1 \cap \mathscr{C}_2 \in C(U)$. Therefore, $(C(U), \cap, \cup)$ is a lattice. □

We notice that $\{\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\}$ is the greatest element of $(C(U), \cap, \cup)$, but $(C(U), \cap, \cup)$ is not a complete lattice necessarily, which is illustrated by the following example.

Example 3.57 Let $U = \{x_1, x_2, x_3\}$, $C_0(U) = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n, ...\} \subseteq C(U)$, and $\mathscr{C}_n = \{\frac{\frac{1}{n}}{x_1} + \frac{\frac{1}{n}}{x_2} + \frac{\frac{1}{n}}{x_3}\}$. By Definition 3.50, it follows that $\bigcap C_0(U) = \{\frac{0}{x_1} + \frac{0}{x_2} + \frac{0}{x_3}\}$. It is obvious that $\bigcap C_0(U) \notin C(U)$. Consequently, $(C(U), \leq)$ is not an intersection structure.

Proposition 3.58 *Let U be a non-empty universe of discourse. Then $(U, C(U) \cup \{\emptyset\})$ is a topological space.*

Proof. Straightforward from Definitions 3.47 and 3.50. □

At the end of this subsection, we provide two roughness measures of fuzzy sets as follows.

Definition 3.59 *Let $(U, \mathscr{C})$ be a fuzzy covering approximation space, and $X \subseteq U$. Then the roughness measure $\mu_{\mathscr{C}}(X)$ regarding $\mathscr{C}$ is defined as*

$$\mu_{\mathscr{C}}(X) = 1 - \frac{|\underline{X}_{\mathscr{C}}|}{|\overline{X}_{\mathscr{C}}|},$$

where $|\underline{X}_{\mathscr{C}}| = \sum_{x \in U} \underline{X}_{\mathscr{C}}(x)$ and $|\overline{X}_{\mathscr{C}}| = \sum_{x \in U} \overline{X}_{\mathscr{C}}(x)$.

Definition 3.60 *Let $(U,\mathscr{C})$ be a fuzzy covering approximation space, and $X \subseteq U$. Then the $\alpha\beta-$roughness measure $\mu^{\alpha\beta}_{\mathscr{C}}(X)$ with respect to $\mathscr{C}$ is defined as*

$$\mu^{\alpha\beta}_{\mathscr{C}}(X) = 1 - \frac{|\underline{X}^{\alpha}_{\mathscr{C}}|}{|\overline{X}^{\beta}_{\mathscr{C}}|},$$

where $\underline{X}^{\alpha}_{\mathscr{C}} = \{x|\underline{X}_{\mathscr{C}}(x) > \alpha, x \in U\}$, $\overline{X}^{\beta}_{\mathscr{C}} = \{x|\overline{X}_{\mathscr{C}}(x) > \beta, x \in U\}$ and $|\cdot|$ means the cardinality of the set.

An example is employed to illustrate Definitions 3.59 and 3.60 as follows.

Example 3.61 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C} = \{C_{21}, C_{22}, C_{23}\}$, and $X = \frac{0.2}{x_1} + \frac{0.5}{x_2} + \frac{0.6}{x_3} + \frac{0.1}{x_4}$. According to Definition 3.8, we have that $\underline{X}_{\mathscr{C}} = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0}{x_4}$ and $\overline{X}_{\mathscr{C}} = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.5}{x_4}$. It follows that $\mu_{\mathscr{C}}(X) = 1 - \frac{0.2+0.4+0.5}{0.2+0.4+0.5+0.5} = 0.3125$. Furthermore, it is obvious that $\underline{X}^{\alpha}_{\mathscr{C}} = \{x_3\}$ and $\overline{X}^{\beta}_{\mathscr{C}} = \{x_2, x_3, x_4\}$ by taking $\alpha = 0.4$ and $\beta = 0.2$. Subsequently, it follows that $\mu^{\alpha\beta}_{\mathscr{C}}(X) = 1 - \frac{|\{x_3\}|}{|\{x_2,x_3,x_4\}|} = \frac{2}{3}$.

4 Consistent functions for fuzzy covering information systems

In [24], Wang et al. proposed the concept of consistent functions for attribute reductions of covering information systems. But so far we have not seen the similar work on fuzzy covering information systems. In this section, we introduce the concepts of consistent functions, the fuzzy covering mappings and inverse fuzzy covering mappings based on fuzzy coverings and examine their basic properties. Additionally, several examples are employed to illustrate our proposed notions.

As a generalization of the concept of consistent functions given in Definition 2.6, we introduce the notion of consistent functions for constructing attribute reducts of fuzzy covering information systems.

Definition 4.1 *Let f be a mapping from U_1 to U_2, $\mathscr{C}$=$\{C_1, C_2, ..., C_N\}$ a fuzzy covering of U_1, and $[x]$ the block of $U_1/IND(f)$ which contains x, where $U_1/IND(f)$ stands for the blocks of partition of U_1 by an equivalence relation $IND(f)$ based on f. If $C_i(y) = C_i(z)$ $(1 \leq i \leq N)$ for any $y, z \in [x]$, then f is called a consistent function with respect to $\mathscr{C}$.*

Unless stated otherwise, we take the equivalence relation $IND(f) = \{(x, y)|f(x) = f(y), x, y \in U_1\}$ and $[x] = \{y \in U_1|f(x) = f(y), x, y \in U_1\}$ when applying Definition 4.1 in this work. Particularly, it is clear that our proposed function is the same as the consistent function in [24] when the membership degree for any $x \in U_1$ has its value only from the set $\{0, 1\}$. Thereby, the proposed model can be viewed as an extension of that given in [24].

An example is employed to illustrate the concept of consistent functions in the following.

Example 4.2 Consider the fuzzy covering approximation space $(U_1, \mathscr{C}_1)$ in Example 3.5. Then, we take $U_2 = \{y_1, y_2\}$ and define a mapping $f : U_1 \longrightarrow U_2$ as

$$f(x_1) = f(x_3) = y_1;\, f(x_2) = f(x_4) = y_2.$$

Obviously, f is a consistent function with respect to $\mathscr{C}_1$.

Now we investigate the relationship between Definitions 2.7 and 4.1. If $\{R_x|x \in U_1\}$ is a fuzzy covering of U_1, where $R_x(y) = R(x, y)$ for any $x, y \in U_1$, then we can express Definition 2.7 as follows: let U_1 and U_2 be two universes, f a mapping from U_1 to U_2, $R \in \mathscr{F}(U_1 \times U_1)$, and $[x]_f = \{y \in U_1 | f(x) = f(y)\}$, $\{[x]_f|x \in U_1\}$ a partition on U_1. For any $x, y \in U_1$, if $R(x, v) = R(x, t)$ for any two pairs $(x, v), (x, t) \in [x]_f \times [y]_f$, then f is said to be consistent with respect to R. Consequently, the consistent function given in Definition 2.7 is the same as our proposed model if $\{R_x|x \in U_1\}$ is a fuzzy covering of U_1 and $IND(f) = \{(x, y)| f(x) = f(y), x, y \in U_1\}$.

In the following, we investigate some conditions under which $\{R_x|x \in U_1\}$ is a fuzzy covering of U_1.

Corollary 4.3 *Let R be a fuzzy relation on U_1. Then*

(1) *if R is α-reflexive, then $\{R_x|x \in U_1\}$ is a fuzzy covering of U_1, where $1 > \alpha > 0$;*

(2) *if R is reflexive, then $\{R_x|x \in U_1\}$ is a fuzzy covering of U_1;*

(3) *if R is a fuzzy similarity, then $\{R_x|x \in U_1\}$ is a fuzzy covering of U_1;*

(4) *if R is a fuzzy equivalence, then $\{R_x|x \in U_1\}$ is a fuzzy covering of U_1.*

Proof. (1) If R is α-reflexive, then $R_x(x) \geq \alpha$ for any $x \in U_1$. It follows that $(\bigcup\{R_x|x \in U_1\})(y) \geq \alpha$ for any $y \in U_1$. Therefore, $\{R_x|x \in U_1\}$ is a fuzzy covering of U_1.

(2) If R is reflexive, then $R_x(x) = 1$ for any $x \in U_1$. It implies that $\bigcup\{R_x|x \in U_1\} = U_1$. Therefore, $\{R_x|x \in U_1\}$ is a fuzzy covering of U_1.

(3), (4) The proof is similar to that in Corollary 4.3(2). □

Additionally, we can construct a fuzzy relation by a fuzzy covering.

Corollary 4.4 *Let f be a mapping from U_1 to U_2, $\mathscr{C}=\{C_1, C_2, ..., C_N\}$ a fuzzy covering of U_1, $R_x = C_x$ for any $x \in U_1$, and $\alpha = min\{C_x(x)|x \in U_1\}$. Then*

(1) *R is a α-reflexive relation;*

(2) *R is symmetric if $C_x(x) = C_y(y)$ for any $x, y \in U_1$;*

(3) *R is transitive;*

(4) *R is a fuzzy equivalence relation if $\alpha = 1$.*

Proof. Straightforward from Definition 3.1. □

By Corollaries 4.3 and 4.4, it is clear that there exists a relationship between a fuzzy relation and a fuzzy covering. Since both Wang's model [24] and our proposed function are based on a fuzzy relation and a fuzzy covering, respectively, by Corollaries 4.3 and 4.4, we can establish the relationship between Definitions 2.7 and 4.1.

By means of Zadeh's extension principle, we propose the concepts of the fuzzy covering mapping and inverse fuzzy covering mapping.

Definition 4.5 *Let $S_1 = (U_1, \mathscr{C}_1)$ and $S_2 = (U_2, \mathscr{C}_2)$ be fuzzy covering approximation spaces, and f a surjection from U_1 to U_2, f induces a mapping from $\mathscr{C}_1$ to $\mathscr{C}_2$ and a mapping from $\mathscr{C}_2$ to $\mathscr{C}_1$, that is*

$$\hat{f} : \mathscr{C}_1 \longrightarrow \mathscr{C}_2, C \mapsto \hat{f}(C) \in \mathscr{C}_2, \forall C \in \mathscr{C}_1;$$

$$\hat{f}(C)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C(x), & f^{-1}(y) \neq \emptyset; \\ 0, & f^{-1}(y) = \emptyset; \end{cases}$$

$$\hat{f}^{-1} : \mathscr{C}_2 \longrightarrow \mathscr{C}_1, T \mapsto \hat{f}^{-1}(T) \in \mathscr{C}_1, \forall T \in \mathscr{C}_2;$$

$$f^{-1}(T)(x) = T(f(x)), x \in U_1.$$

Then $\hat{f}$ and $\hat{f}^{-1}$ are called the fuzzy covering mapping and the inverse fuzzy covering mapping induced by f, respectively. In convenience, we denote $\hat{f}$ and $\hat{f}^{-1}$ as f and f^{-1}, respectively.

By Definition 4.5, we observe that $\hat{f}$ and $\hat{f}^{-1}$ will be reduced to Definition 4.1 in [24] if the membership degree takes values from the set $\{0, 1\}$. The following theorem discusses the problem of fuzzy set operations under a consistent function f.

Theorem 4.6 *Let f be a mapping from U_1 to U_2, $\mathscr{C} = \{C_1, C_2, ..., C_N\}$ a fuzzy covering of U_1, and $C_i, C_j \in \mathscr{C}$. Then*

(1) *$f(C_i \cap C_j) \subseteq f(C_i) \cap f(C_j)$;*

(2) *$f(C_i \cup C_j) = f(C_i) \cup f(C_j)$;*

(3) *If* f *is a consistent function with respect to* $\mathscr{C}$, *then* $f(C_i \cap C_j) = f(C_i) \cap f(C_j)$.

Proof. (1) By Definition 4.5, we obtain that $f(C_i \cap C_j)(f(x)) = 0$ when $(C_i \cap C_j)(x) = 0$ for $x \in U_1$. Moreover, by Definition 4.5, it follows that $f(C_i \cap C_j)(y) = \bigvee_{x' \in f^{-1}(y)}(C_i \cap C_j)(x') = \bigvee_{x' \in f^{-1}(y)}(C_i(x') \wedge C_j(x')) \leq \bigvee_{x' \in f^{-1}(y)} C_i(x') \wedge \bigvee_{x' \in f^{-1}(y)} C_j(x') = (f(C_i) \cap f(C_j))(y)$. Consequently, $f(C_i \cap C_j) \subseteq f(C_i) \cap f(C_j)$.

(2) According to Definition 4.5, we have that $f(C_i \cup C_j)(f(x)) = 0$ when $(C_i \cup C_j)(x) = 0$ for $x \in U_1$. Furthermore, by Definition 4.5, it follows that $f(C_i \cup C_j)(y) = \bigvee_{x' \in f^{-1}(y)}(C_i \cup C_j)(x') = \bigvee_{x' \in f^{-1}(y)}(C_i(x') \vee C_j(x')) = \bigvee_{x' \in f^{-1}(y)} C_i(x') \vee \bigvee_{x' \in f^{-1}(y)} C_j(x') = (f(C_i) \cup f(C_j))(y)$. Therefore, $f(C_i \cup C_j) = f(C_i) \cup f(C_j)$.

(3) By Theorem 4.6(1), it is obvious that $f(C_i \cap C_j) \subseteq f(C_i) \cap f(C_j)$. So we only need to prove that $f(C_i) \cap f(C_j) \subseteq f(C_i \cap C_j)$. Suppose that $y \in U_2$, there exists $x \in U_1$ such that $f(x) = y$. Based on Definitions 4.1 and 4.5, we have that $(f(C_i) \cap f(C_j))(y) = \bigvee_{x' \in f^{-1}(y)} C_i(x') \wedge \bigvee_{x' \in f^{-1}(y)} C_j(x') = C_i(x') \wedge C_j(x') \subseteq \bigvee_{x' \in f^{-1}(y)}(C_i(x') \wedge C_j(x')) = f(C_i \cap C_j)(y)$. Thereby, $f(C_i \cap C_j) = f(C_i) \cap f(C_j)$. □

Theorem 4.6 shows that the mapping f preserves some fuzzy set operations, especially it preserves the intersection operation of fuzzy sets if f is consistent.

To illustrate Theorem 4.6, we give an example below.

Example 4.7 Consider $S = (U_1, \mathscr{C}_1)$ in Example 3.5 and the consistent function f in Example 4.2. Then we observe that $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$, $f(C_1 \cap C_3) = f(C_1) \cap f(C_3)$ and $f(C_2 \cap C_3) = f(C_2) \cap f(C_3)$.

By Theorem 4.6, we obtain the following corollary.

Corollary 4.8 *Let* f *be a mapping from* U_1 *to* U_2, *and* $\mathscr{C}=\{C_1, C_2, ..., C_N\}$ *a fuzzy covering of* U_1. *If* f *is a consistent function with respect to* $\mathscr{C}$, *then* $f(\bigcap_{i=1}^{N} C_i) = \bigcap_{i=1}^{N} f(C_i)$.

Subsequently, we investigate the properties of the inverse mapping of a consistent function.

Theorem 4.9 *Let* f *be a mapping from* U_1 *to* U_2, $\mathscr{C}=\{C_1, C_2, ..., C_N\}$ *a fuzzy covering of* U_1, *and* $C_i \in \mathscr{C}$. *Then*

(1) $C_i \subseteq f^{-1}(f(C_i))$;

(2) *If* f *is a consistent function with respect to* $\mathscr{C}$, *then* $f^{-1}(f(C_i)) = C_i$.

Proof. (1) According to Definition 4.5, we have that $f^{-1}(f(C_i))(x) = f(C_i)(f(x))$. Taking $y = f(x)$, it follows that $f(C_i)(f(x)) = f(C_i)(y) = \bigvee_{x' \in f^{-1}(y)} C_i(x') \geq C_i(x)$. Therefore, $C_i \subseteq f^{-1}(f(C_i))$.

(2) By Definition 4.5, we see that $f^{-1}(f(C_i(x))) = f(C_i)(f(x))$. Assume that $y = f(x)$, it follows that $f(C_i)(f(x)) = f(C_i)(y) = \bigvee_{x' \in f^{-1}(y)} C_i(x')$. According to Definitions 4.1 and 4.5, it implies that $C_i(x') = C_i(x)$ for any $x' \in f^{-1}(y)$. Consequently, $\bigvee_{x' \in f^{-1}(y)} C_i(x') = C_i(x)$. Hence, $f^{-1}(f(C_i))(x) = C_i(x)$. Thereby, $C_i = f^{-1}(f(C_i))$. □

We give an example to illustrate Theorem 4.9 in the following.

Example 4.10 Consider $S = (U_1, \mathscr{C}_1)$ in Example 3.5 and the consistent function f in Example 4.2. Then we see that $f^{-1}(f(C_1))(x_i) = C_1(x_i)$, $f^{-1}(f(C_2))(x_i) = C_2(x_i)$, and $f^{-1}(f(C_3))(x_i) = C_3(x_i), i = 1, 2, 3, 4$. Therefore, $f^{-1}(f(C_1)) = C_1$, $f^{-1}(f(C_2)) = C_2$, and $f^{-1}(f(C_3)) = C_3$.

By Theorem 4.9, we have the following corollary.

Corollary 4.11 *Let f be a mapping from U_1 to U_2, and $\mathscr{C}=\{C_1, C_2, ..., C_N\}$ a fuzzy covering of U_1. If f is a consistent function with respect to $\mathscr{C}$, then $f^{-1}(f(\bigcap_{i=1}^{N} C_i)) = \bigcap_{i=1}^{N} C_i$.*

We also explore the properties of a consistent function on a family of fuzzy coverings.

Theorem 4.12 *Let f be a mapping from U_1 to U_2, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U_1)$. If f is a consistent function with respect to $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively, then f is consistent with respect to $\mathscr{C}_1 \bigcap \mathscr{C}_2$.*

Proof. Based on Definition 4.1, we have that $C_{ix}(y) = C_{ix}(z)$ for any $y, z \in [x]$, where $i = 1, 2$. It follows that $C_{1x}(y) \wedge C_{2x}(y) = C_{1x}(z) \wedge C_{2x}(z)$ for any $y, z \in [x]$. Hence, $(C_{1x} \cap C_{2x})(y) = (C_{1x} \cap C_{2x})(z)$ for any $y, z \in [x]$. Therefore, f is consistent with respect to $\mathscr{C}_1 \bigcap \mathscr{C}_2$. □

The following example is employed to illustrate Theorem 4.12.

Example 4.13 Consider $S = (U_1, \Delta)$ in Example 3.7. We take $U_2 = \{y_1, y_2\}$ and define a mapping $f : U_1 \to U_2$ as follows:

$$f(x_1) = f(x_2) = y_1, f(x_3) = f(x_4) = y_2.$$

It is obvious that f is a consistent function with respect to $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively. By Definition 4.1, we observe that f is consistent with respect to $\mathscr{C}_1 \cap \mathscr{C}_2$.

Based on Theorem 4.12, we obtain the following corollary.

Corollary 4.14 *Let $\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m \in C(U_1)$, and f a mapping from U_1 to U_2. If f is a consistent function with respect to any $\mathscr{C}_i$ $(1 \leq i \leq m)$, then f is consistent with respect to $\bigcap_{i=1}^{m} \mathscr{C}_i$.*

Now, we introduce two concepts for fuzzy covering approximation spaces.

Definition 4.15 *Let f be a mapping from U_1 to U_2, $\mathscr{C}_1 = \{C_{11}, C_{12}, ..., C_{1N}\} \in C(U_1)$, and $\mathscr{C}_2 = \{T_{21}, T_{22}, ..., T_{2M}\} \in C(U_2)$. Then $f(\mathscr{C}_1)$ and $f(\mathscr{C}_2)$ are defined by*

$$\begin{aligned} f(\mathscr{C}_1) &= \{f(C_{1i}), C_{1i} \in \mathscr{C}_1, 1 \leq i \leq N\}; \\ f^{-1}(\mathscr{C}_2) &= \{f^{-1}(T_{2j}), T_{2j} \in \mathscr{C}_2, 1 \leq j \leq M\}. \end{aligned}$$

Theorem 4.16 *Let U be a non-empty universe of discourse, and $\mathscr{C} \in C(U)$. If f is a consistent function with respect to $\mathscr{C}$, then $f^{-1}(f(\mathscr{C})) = \mathscr{C}$.*

Proof. By Theorem 4.9, it follows that $f^{-1}(f(C_i)) = C_i$ for any $C_i \in \mathscr{C}$. Therefore, $f^{-1}(f(\mathscr{C})) = \mathscr{C}$. □

Obviously, Examples 3.5 and 4.2 can illustrate Theorem 4.16. Then we get the following corollary.

Corollary 4.17 *Let $\mathscr{C}_i \in C(U)$, and $\Delta = \{\mathscr{C}_i | i = 1, 2, ..., m\}$. If f is a consistent function with respect to any $\mathscr{C}_i \in \Delta$, then $f^{-1}(f(\bigcap \Delta)) = \bigcap \Delta$.*

At the end of this section, we discuss the fuzzy covering operations under a consistent function.

Theorem 4.18 *Let f be a mapping from U_1 to U_2, and $\mathscr{C}_1, \mathscr{C}_2 \in C(U_1)$. Then we have*

(1) *$f(\mathscr{C}_1 \bigcap \mathscr{C}_2) \subseteq f(\mathscr{C}_1) \bigcap f(\mathscr{C}_2)$;*

(2) *If f is a consistent function with respect to $\mathscr{C}_1$ and $\mathscr{C}_2$, respectively, then $f(\mathscr{C}_1 \bigcap \mathscr{C}_2) = f(\mathscr{C}_1) \bigcap f(\mathscr{C}_2)$.*

Proof. (1) According to Definitions 4.1 and 4.5, it is obvious that $f(\mathscr{C}_1 \bigcap \mathscr{C}_2) \subseteq f(\mathscr{C}_1) \bigcap f(\mathscr{C}_2)$.

(2) Evidently, we only need to prove that $f(\mathscr{C}_1) \bigcap f(\mathscr{C}_2) \subseteq f(\mathscr{C}_1 \bigcap \mathscr{C}_2)$. Assume that C_x is the minimal element containing x in $\mathscr{C}_1 \bigcap \mathscr{C}_2$, C_{1x} is the minimal element containing x in $Cov(\mathscr{C}_1)$, and C_{2x} is the minimal element containing x in $Cov(\mathscr{C}_2)$ for any $x \in U_1$. By Definition 3.50, it follows that $C_x = C_{1x} \bigcap C_{2x}$. According to Theorem 4.12, it implies that f is a consistent function with respect to $\mathscr{C}_1 \bigcap \mathscr{C}_2$. Consequently, we obtain that $f(C_x) = f(C_{1x}) \cap f(C_{2x})$. By Definition 3.1, we have that $C_x(x) > 0$ for any $x \in U_1$. It follows that $f(C_x)(f(x)) > 0$. Hence, $(f(C_{1x}) \cap f(C_{2x}))(f(x)) > 0$. Suppose that $f(C_{1x}) \cap f(C_{2x})$ is not the minimal subset containing $f(x)$ in $f(\mathscr{C}_1 \bigcap \mathscr{C}_2)$. Then there exists $x_0 \in U_1$ such that $f(C_{ix_0})(f(x)) > 0$ and $f(C_{1x}) \cap f(C_{2x}) \cap f(C_{ix_0}) \subset f(C_{1x}) \cap f(C_{2x})$, it means that $(f(C_{1x}) \cap f(C_{2x}) \cap f(C_{ix_0}))(f(x)) > 0$. Thereby, there exist u, v and w such that $C_{1x}(u) > 0, C_{2x}(v) > 0, C_{ix_0}(w) > 0$ and $f(u) = f(v) = f(w) = f(x)$. According to Theorem 4.6, we have that $f(C_{1x}) \cap f(C_{2x}) = f(C_{1x} \cap C_{2x}) \subseteq f(C_{ix_0})$

and $f(C_{1x}) \cap f(C_{2x}) \cap f(C_{ix_0}) = f(C_{1x}) \cap f(C_{2x})$, it implies that $f(C_{1x}) \cap f(C_{2x})$ is the minimal subset containing $f(x)$ in $f(\mathscr{C}_1 \bigcap \mathscr{C}_2)$. Based on the above statement, it follows that $f(\mathscr{C}_1) \bigcap f(\mathscr{C}_2) \subseteq f(\mathscr{C}_1 \bigcap \mathscr{C}_2)$. Therefore, $f(\mathscr{C}_1 \bigcap \mathscr{C}_2) = f(\mathscr{C}_1) \bigcap f(\mathscr{C}_2)$. □

Based on Theorem 4.18, we have the following corollary.

Corollary 4.19 *Let $\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m$ be fuzzy coverings of U_1, and f a mapping from U_1 to U_2. If f is a consistent function with respect to $\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m$, respectively, then $f(\bigcap_{i=1}^m \mathscr{C}_i) = \bigcap_{i=1}^m f(\mathscr{C}_i)$.*

5 Data compressions of fuzzy covering information systems and dynamic fuzzy covering information systems

In this section, we further investigate data compressions of fuzzy covering information systems and dynamic fuzzy covering information systems.

5.1 Data compression of fuzzy covering information systems

In this subsection, the concepts of an induced fuzzy covering information system and homomorphisms between fuzzy covering information systems are introduced for data compression of the fuzzy covering information system. Then the algorithm of constructing attribute reducts of fuzzy covering information systems is provided. An example is finally employed to illustrate the proposed concepts and algorithm.

Definition 5.1 *Let f be a surjection from U_1 to U_2, Δ_1=$\{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$ a family of fuzzy coverings of U_1, and $f(\Delta_1)$=$\{f(\mathscr{C}_1), f(\mathscr{C}_2), ..., f(\mathscr{C}_m)\}$. Then (U_1, Δ_1) is referred to as a fuzzy covering information system and $(U_2, f(\Delta_1))$ is called the f-induced fuzzy covering information system of (U_1, Δ_1).*

Definition 5.1 shows that we can induce a new fuzzy covering information system under a surjection.

Based on Definitions 4.1 and 5.1, we propose the notion of a homomorphism between two fuzzy covering information systems.

Definition 5.2 *Let f be a surjection from U_1 to U_2, Δ_1=$\{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$ a family of fuzzy coverings of U_1, and $f(\Delta_1)$=$\{f(\mathscr{C}_1), f(\mathscr{C}_2), ..., f(\mathscr{C}_m)\}$. If f is consistent with respect to any $\mathscr{C}_i \in \Delta_1$ $(1 \leq i \leq m)$ on U_1, then f is called a homomorphism from (U_1, Δ_1) to $(U_2, f(\Delta_1))$.*

We provide the concept of reducts of fuzzy covering information systems in the following.

Definition 5.3 *Let (U_1, Δ_1) be a fuzzy covering information system, and $\mathscr{C}_i \in \Delta_1$ $(1 \leq i \leq m)$. If $\bigcap\{\Delta_1 - \mathscr{C}_i\} = \bigcap \Delta_1$, then $\mathscr{C}_i$ is called superfluous. Otherwise, $\mathscr{C}_i$ is called indispensable. The collection of all indispensable elements in Δ_1, denoted as Core(Δ_1), is called the core of Δ_1. $P \subseteq \Delta_1$ is called a reduct of Δ_1 if P satisfies: $\bigcap P = \bigcap \Delta_1$ and $\bigcap\{P - \mathscr{C}\} \neq \bigcap \Delta_1$ for any $\mathscr{C} \in P$.*

Now we present the following theorem which shows that the reducts of fuzzy covering information systems can be preserved under a homomorphism.

Theorem 5.4 *Let f be a surjection from U_1 to U_2, Δ_1=$\{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$ a family of fuzzy coverings of U_1, and $f(\Delta_1)$=$\{f(\mathscr{C}_1), f(\mathscr{C}_2), ..., f(\mathscr{C}_m)\}$. If f is a homomorphism from (U_1, Δ_1) to $(U_2, f(\Delta_1))$, then $P \subseteq \Delta_1$ is a reduct of Δ_1 if and only if $f(P)$ is a reduct of $f(\Delta_1)$.*

Proof. Suppose P is a reduct of Δ_1. It follows that $\bigcap P = \bigcap \Delta_1$. Hence, $f(\bigcap P) = f(\bigcap \Delta_1)$. Then we obtain that $\bigcap f(P) = \bigcap f(\Delta_1)$ since f is a homomorphism from (U_1, Δ_1) to $(U_2, f(\Delta_1))$. Assume that there exists $\mathscr{C} \in P$ such that $\bigcap(f(P) - f(\mathscr{C})) = \bigcap f(P)$. It implies that $\bigcap(f(P) - f(\mathscr{C})) = \bigcap f(P - \mathscr{C})$. Hence, we see that $\bigcap f(\Delta_1) = \bigcap f(P - \mathscr{C})$. It follows that $f^{-1}(\bigcap f(\Delta_1)) = f^{-1}(\bigcap f(P - \mathscr{C}))$. We obtain that $\bigcap \Delta_1 = \bigcap(P - \mathscr{C})$, which contradicts that P is a reduct of Δ_1. So $f(P)$ is a reduct of $f(\Delta_1)$.

On the other hand, we assume that $f(P)$ is a reduct of $f(\Delta_1)$. It follows that $\bigcap f(\Delta_1) = \bigcap f(P)$. Since f is a homomorphism from (U_1, Δ_1) to $(U_2, f(\Delta_1))$, we obtain that $f(\bigcap \Delta_1) = f(\bigcap P)$. It implies that $\bigcap \Delta_1 = \bigcap P$. Assume that there exists $\mathscr{C} \in P$ satisfying $\bigcap \Delta_1 = \bigcap(P - \mathscr{C})$, it follows that $f(\bigcap \Delta_1) = f(\bigcap(P - \mathscr{C}))$. Obviously, $\bigcap f(\Delta_1) = \bigcap f(P - \mathscr{C}) = \bigcap(f(P) - f(\mathscr{C}))$, which is a contradiction. Therefore, $P \subseteq \Delta_1$ is a reduct of Δ_1. $\square$

By Theorem 5.4, we obtain the following corollary.

Corollary 5.5 *Let f be a surjection from U_1 to U_2, Δ_1=$\{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$ a family of fuzzy coverings of U_1, and $f(\Delta_1)$=$\{f(\mathscr{C}_1), f(\mathscr{C}_2), ..., f(\mathscr{C}_m)\}$. If f is a homomorphism from (U_1, Δ_1) to $(U_2, f(\Delta_1))$, then*

(1) *$\mathscr{C}$ is indispensable in Δ_1 if and only if $f(\mathscr{C})$ is indispensable in $f(\Delta_1)$;*

(2) *$\mathscr{C}$ is superfluous in Δ_1 if and only if $f(\mathscr{C})$ is superfluous in $f(\Delta_1)$;*

(3) *The image of the core of Δ_1 is the core of $f(\Delta_1)$, and the inverse image of the core of $f(\Delta_1)$ is the core of the original image.*

Proof. Straightforward from Definition 5.3 and Theorem 5.4. $\square$

From Corollary 5.5, we see that the attribute reductions of the original fuzzy covering information system and image system are equivalent to each other under the condition of a homomorphism.

Definition 5.6 *Let $(U_1, \mathscr{C}_1)$ be a fuzzy covering approximation space, the equivalence relation $R_{\mathscr{C}_1} = \{(x,y)|C_x = C_y, x, y \in U_1\}$, and $U_1/R_{\mathscr{C}_1} = \{R_{\mathscr{C}_1}(x)|x \in U_1\}$. Then $U_1/R_{\mathscr{C}_1}$ is called the partition based on $\mathscr{C}_1$.*

For the sake of convenience, we denote $U_1/R_{\mathscr{C}_1}$ as $U_1/\mathscr{C}_1$ simply.

Following, we employ Table 2 to show the partition based on each fuzzy covering for the fuzzy covering information system (U_1, Δ_1), where P_{ix_j} stands for the block containing x_j in the partition $U_1/R_{\mathscr{C}_i}$. It is easy to see that $P_{\Delta_1 x_j} = \bigcap_{1 \leq i \leq m} P_{ix_j}$, where $P_{\Delta_1 x_j}$ denotes the block containing x_j in the partition based on Δ_1.

Subsequently, we propose the main feature of the algorithm to construct attribute reducts of fuzzy covering information systems. It shows how to construct a homomorphism and compress a large-scale information system into a small one under the condition of the homomorphism.

Algorithm 5.7 *Let $U_1 = \{x_1, ..., x_n\}$, and Δ_1=$\{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$ a family of fuzzy coverings of U_1. Then*

Step 1. Input the fuzzy covering information system (U_1, Δ_1);

Step 2. Computing the partition $U_1/\mathscr{C}_i$ $(1 \leq i \leq m)$ and obtain $U_1/\Delta_1 = \{C_i | 1 \leq i \leq K\}$;

Step 3. Define $f : U_1 \to U_2$ as follows: $f(x) = y_l$, $x \in C_l$, where $1 \leq l \leq K$ and $U_2 = \{y_1, y_2, ..., y_K\}$;

Step 4. Compute $f(\Delta_1)$=$\{f(\mathscr{C}_1), f(\mathscr{C}_2), ..., f(\mathscr{C}_m)\}$ and obtain $(U_2, f(\Delta_1))$;

Step 5. Construct attribute reducts of $(U_2, f(\Delta_1))$ and obtain a reduct $\{f(\mathscr{C}_{i1}), f(\mathscr{C}_{i2}), ..., f(\mathscr{C}_{ik})\}$;

Step 6. Obtain a reduct $\{\mathscr{C}_{i1}, \mathscr{C}_{i2}, ..., \mathscr{C}_{ik}\}$ of (U_1, Δ_1) and output the results.

Remark. In Example 5.1 [24], Wang et al. obtained the partition U_1/Δ_1 by only computing Δ_x for any $x \in U_1$. But we get U_1/Δ_1 by computing $U_1/\mathscr{C}_i$ for any $\mathscr{C}_i \in \Delta_1$ in Algorithm 5.7. By using the proposed approach, we can compress the dynamic fuzzy covering information system on the basis of data compression of the original system with lower time complexity, which is illustrated in Subsection 5.2.

Now, we employ a car evaluation problem to illustrate Algorithm 5.7.

Example 5.8 Suppose that $U_1 = \{x_1, x_2, ..., x_8\}$ is a set of eight cars, $C_1 = \{price, structure, size, appearance\}$ is a set of attributes. The domains of $price$, $structure$, $size$ and $appearance$ are $\{high, middle, low\}$,

$\{excellent, ordinary, poor\}$, $\{big, middle, small\}$ and $\{beautiful, fair, ugly\}$, respectively. In this example, we do not list their evaluation reports for simplicity. According to the four specialists' evaluation reports, we obtain the following fuzzy coverings of U_1 as $\Delta_1 = \{\mathscr{C}_{price}, \mathscr{C}_{structure}, \mathscr{C}_{size}, \mathscr{C}_{appearance}\}$, $\mathscr{C}_{price}, \mathscr{C}_{structure}, \mathscr{C}_{size}$ and $\mathscr{C}_{appearance}$ are based on *price*, *structure*, *size* and *appearance*, respectively, where

$$\begin{aligned}
\mathscr{C}_{price} &= \{\frac{1}{x_1}+\frac{1}{x_2}+\frac{0.5}{x_3}+\frac{1}{x_4}+\frac{0.5}{x_5}+\frac{1}{x_6}+\frac{1}{x_7}+\frac{1}{x_8}, \frac{0.5}{x_1}+\frac{0.5}{x_2}+\frac{0.5}{x_3}+\frac{1}{x_4}+\frac{0.5}{x_5}+\frac{0.5}{x_6}+\frac{1}{x_7}+\frac{1}{x_8},\\
&\quad \frac{0}{x_1}+\frac{0}{x_2}+\frac{1}{x_3}+\frac{0.5}{x_4}+\frac{1}{x_5}+\frac{1}{x_6}+\frac{0.5}{x_7}+\frac{0.5}{x_8}\};\\
\mathscr{C}_{structure} &= \{\frac{0}{x_1}+\frac{0}{x_2}+\frac{1}{x_3}+\frac{0}{x_4}+\frac{1}{x_5}+\frac{0}{x_6}+\frac{0}{x_7}+\frac{0}{x_8}, \frac{1}{x_1}+\frac{1}{x_2}+\frac{0.5}{x_3}+\frac{1}{x_4}+\frac{0.5}{x_5}+\frac{1}{x_6}+\frac{1}{x_7}+\frac{1}{x_8},\\
&\quad \frac{1}{x_1}+\frac{1}{x_2}+\frac{0.5}{x_3}+\frac{0.5}{x_4}+\frac{0.5}{x_5}+\frac{0}{x_6}+\frac{0.5}{x_7}+\frac{0.5}{x_8}\};\\
\mathscr{C}_{size} &= \{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{0}{x_4}+\frac{1}{x_5}+\frac{1}{x_6}+\frac{0}{x_7}+\frac{0}{x_8}, \frac{0.5}{x_1}+\frac{0.5}{x_2}+\frac{1}{x_3}+\frac{0.5}{x_4}+\frac{1}{x_5}+\frac{0.5}{x_6}+\frac{0.5}{x_7}+\frac{0.5}{x_8},\\
&\quad \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}+\frac{0.5}{x_6}+\frac{1}{x_7}+\frac{1}{x_8}\};\\
\mathscr{C}_{appearance} &= \{\frac{1}{x_1}+\frac{1}{x_2}+\frac{0.5}{x_3}+\frac{1}{x_4}+\frac{0.5}{x_5}+\frac{1}{x_6}+\frac{1}{x_7}+\frac{1}{x_8}, \frac{1}{x_1}+\frac{1}{x_2}+\frac{0.5}{x_3}+\frac{1}{x_4}+\frac{0.5}{x_5}+\frac{1}{x_6}+\frac{1}{x_7}+\frac{1}{x_8},\\
&\quad \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}+\frac{0.5}{x_6}+\frac{1}{x_7}+\frac{1}{x_8}\}.
\end{aligned}$$

By Definition 2.8, we see that (U_1, Δ_1) is a fuzzy covering information system. Furthermore, according to Definitions 3.1, 3.6 and 5.6, we obtain the following results:

$$\begin{aligned}
U_1/\mathscr{C}_{price} &= \{\{x_1, x_2\}, \{x_3, x_4, x_5, x_6, x_7, x_8\}\};\\
U_1/\mathscr{C}_{structure} &= \{\{x_1, x_2, x_4, x_7, x_8\}, \{x_3, x_5\}, \{x_6\}\};\\
U_1/\mathscr{C}_{size} &= \{\{x_1, x_2, x_3, x_5, x_6\}, \{x_4, x_7, x_8\}\};\\
U_1/\mathscr{C}_{appearance} &= \{\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}\}.
\end{aligned}$$

The partitions $U_1/\mathscr{C}_{price}, U_1/\mathscr{C}_{structure}, U_1/\mathscr{C}_{size}$ and $U_1/\mathscr{C}_{appearance}$ are shown in Table 3. Then we obtain that $U_1/\Delta_1 = \{\{x_1, x_2\}, \{x_3, x_5\}, \{x_4, x_7, x_8\}, \{x_6\}\}$. Thus we take $U_2 = \{y_1, y_2, y_3, y_4\}$ and define a mapping $f : U_1 \longrightarrow U_2$ as follows:

$$f(x_1) = f(x_2) = y_1;\ f(x_3) = f(x_5) = y_2;\ f(x_4) = f(x_7) = f(x_8) = y_3;\ f(x_6) = y_4.$$

According to the function f, we obtain that $f(\Delta_1) = \{f(\mathscr{C}_{price}), f(\mathscr{C}_{structure}), f(\mathscr{C}_{size}), f(\mathscr{C}_{appearance})\}$,

where

$$\begin{aligned}
f(\mathscr{C}_{price}) &= \{\frac{1}{y_1}+\frac{0.5}{y_2}+\frac{1}{y_3}+\frac{1}{y_4},\frac{0.5}{y_1}+\frac{0.5}{y_2}+\frac{1}{y_3}+\frac{0.5}{y_4},\frac{0}{y_1}+\frac{1}{y_2}+\frac{0.5}{y_3}+\frac{1}{y_4}\};\\
f(\mathscr{C}_{structure}) &= \{\frac{0}{y_1}+\frac{1}{y_2}+\frac{0}{y_3}+\frac{0}{y_4},\frac{1}{y_1}+\frac{0.5}{y_2}+\frac{1}{y_3}+\frac{1}{y_4},\frac{1}{y_1}+\frac{0.5}{y_2}+\frac{0.5}{y_3}+\frac{0}{y_4}\};\\
f(\mathscr{C}_{size}) &= \{\frac{1}{y_1}+\frac{1}{y_2}+\frac{0}{y_3}+\frac{1}{y_4},\frac{0.5}{y_1}+\frac{1}{y_2}+\frac{0.5}{y_3}+\frac{0.5}{y_4},\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{0.5}{y_4}\};\\
f(\mathscr{C}_{appearance}) &= \{\frac{1}{y_1}+\frac{0.5}{y_2}+\frac{1}{y_3}+\frac{1}{y_4},\frac{1}{y_1}+\frac{0.5}{y_2}+\frac{1}{y_3}+\frac{1}{y_4},\frac{1}{y_1}+\frac{1}{y_2}+\frac{1}{y_3}+\frac{0.5}{y_4}\}.
\end{aligned}$$

According to Definition 5.1, we obtain the f-induced fuzzy covering information system $(U_2, f(\Delta_1))$ of (U_1,Δ_1). Clearly, the size of $(U_2, f(\Delta_1))$ is relatively smaller than that of (U_1,Δ_1). Then, by Definitions 5.1, 5.2 and 5.3, we have the following results:

(1) f is a homomorphism from (U_1,Δ_1) to $(U_2, f(\Delta_1))$;

(2) $f(\mathscr{C}_{appearance})$ is superfluous in $f(\Delta_1)$ if and only if $\mathscr{C}_{appearance}$ is superfluous in Δ_1;

(3) $\{f(\mathscr{C}_{price}), f(\mathscr{C}_{structure}), f(\mathscr{C}_{size})\}$ is a reduct of $f(\Delta_1)$ if and only if $\{\mathscr{C}_{price},\mathscr{C}_{structure},\mathscr{C}_{size}\}$ is a reduct of Δ_1.

From Example 5.8, we see that the image system $(U_2, f(\Delta_1))$ has relatively smaller size than the original system (U_1,Δ_1). But their attribute reductions are equivalent to each other under the condition of a homomorphism.

From the practical viewpoint, it may be difficult to construct attribute reducts of a large-scale fuzzy covering information system directly. However, we can compress it into a relatively smaller fuzzy covering information system under the condition of a homomorphism and conduct the attribute reductions on the image system. Therefore, the notion of a homomorphism may provide a more efficient approach to dealing with large-scale fuzzy covering information systems.

5.2 Data compression of dynamic fuzzy covering information systems

In Subsection 5.1, we derive a partition based on each fuzzy covering shown in Table 2, which is useful for data compression of dynamic fuzzy covering information systems. In the following, we discuss how to compress two types of dynamic fuzzy covering information systems by utilizing the compression of the original fuzzy covering information system.

Type 1: Adding a family of fuzzy coverings. By adding a fuzzy covering $\mathscr{C}_{m+1}$ to the fuzzy covering information system (U_1,Δ_1), we obtain the dynamic fuzzy covering information system (U_1,Δ), where

$\Delta = \Delta_1 \cup \{\mathscr{C}_{m+1}\}$. There are three steps to compress the dynamic fuzzy covering information system (U_1, Δ). First, we obtain the partition $U_1/\mathscr{C}_{m+1}$ in the sense of Definition 5.6 and get Table 4 by adding $U_1/\mathscr{C}_{m+1}$ into Table 2. Then we derive the partition U_1/Δ based on $U_1/\mathscr{C}_i$ $(1 \leq i \leq m+1)$. Afterwards, we define the homomorphism f based on U_1/Δ as Example 5.8 and compress (U_1, Δ) into a small-scale fuzzy covering information system $(f(U_1), f(\Delta))$. Furthermore, the same process can be applied to the dynamic fuzzy covering information system when adding a family of fuzzy coverings.

Type 2: Deleting a family of fuzzy coverings. We obtain the dynamic fuzzy covering information system (U_1, Δ) when deleting the fuzzy covering $\mathscr{C}_k \in \Delta_1$, where $\Delta = \Delta_1 - \{\mathscr{C}_k\}$. To compress the dynamic fuzzy covering information system (U_1, Δ), we first derive Table 5 by canceling the partition $U_1/\mathscr{C}_k$ in Table 2. Then we obtain the partition U_1/Δ based on $U_1/\mathscr{C}_i$ $(1 \leq i \leq k-1, k+1 \leq i \leq m)$ and define the homomorphism f as Example 5.8. Afterwards, (U_1, Δ) is compressed into a small-scale fuzzy covering information system $(f(U_1), f(\Delta))$. Moreover, we can compress the dynamic fuzzy covering information system when deleting a family of fuzzy coverings using the same approach.

In practice, it may be very costly or even intractable to construct the compression of the dynamic fuzzy covering information system as the original fuzzy covering information system. Thus the proposed approach based on the compression of the original fuzzy covering information system may provide a more efficient approach to dealing with data compression of dynamic fuzzy covering information systems.

6 Conclusion and further research

In this paper, we have presented some new operations on fuzzy coverings and investigated their properties in detail. Particularly, the lower and upper approximation operations based on fuzzy coverings have been introduced for the fuzzy covering approximation space. Then we have constructed a consistent function for the communication between fuzzy covering information systems, and pointed out that a homomorphism is a special fuzzy covering mapping between the two fuzzy covering information systems. In addition, we have proved that attribute reductions of the original system and image system are equivalent to each other under the condition of a homomorphism. We have also applied the proposed approach to attribute reductions of fuzzy covering information systems and dynamic fuzzy covering information systems.

In future, we will further study the fuzzy covering information systems by extending the covering

rough sets and apply the proposed method to feature selections of fuzzy covering information systems. Furthermore, we will discuss the data compression of dynamic relation information systems and dynamic fuzzy relation information systems. Especially, we will apply an incremental updating scheme to maintain the compression dynamically and avoid unnecessary computations by utilizing the compression of the original system.

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Table 1: An incomplete information system.

U	*structure*	*color*	*price*
x_1	*bad*	*good*	*low*
x_2	$*$	*good*	*high*
x_3	*good*	*bad*	*high*
x_4	*bad*	*bad*	$*$
x_5	*good*	$*$	*low*
x_6	$*$	*bad*	$*$

Table 2: The partitions based on each fuzzy covering $\mathscr{C}_i$ $(1 \le i \le m)$ and Δ_1, respectively.

U_1	$\mathscr{C}_1$	$\mathscr{C}_2$	.	.	.	$\mathscr{C}_m$	Δ_1
x_1	P_{1x_1}	P_{2x_1}	.	.	.	P_{mx_1}	$P_{\Delta_1 x_1}$
x_2	P_{1x_2}	P_{2x_2}	.	.	.	P_{mx_2}	$P_{\Delta_1 x_2}$
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
x_n	P_{1x_n}	P_{2x_2}	.	.	.	P_{mx_n}	$P_{\Delta_1 x_n}$

Table 3: The partitions based on $\mathscr{C}_{prize}, \mathscr{C}_{structure}, \mathscr{C}_{size}, \mathscr{C}_{appearance}$ and Δ_1, respectively.

U_1	$\mathscr{C}_{price}$	$\mathscr{C}_{structure}$	$\mathscr{C}_{size}$	$\mathscr{C}_{appearance}$	Δ_1
x_1	$\{x_1, x_2\}$	$\{x_1, x_2, x_4, x_7, x_8\}$	$\{x_1, x_2, x_3, x_5, x_6\}$	U_1	$\{x_1, x_2\}$
x_2	$\{x_1, x_2\}$	$\{x_1, x_2, x_4, x_7, x_8\}$	$\{x_1, x_2, x_3, x_5, x_6\}$	U_1	$\{x_1, x_2\}$
x_3	$\{x_3, x_4, x_5, x_6, x_7, x_8\}$	$\{x_3, x_5\}$	$\{x_1, x_2, x_3, x_5, x_6\}$	U_1	$\{x_3, x_5\}$
x_4	$\{x_3, x_4, x_5, x_6, x_7, x_8\}$	$\{x_1, x_2, x_4, x_7, x_8\}$	$\{x_4, x_7, x_8\}$	U_1	$\{x_4, x_7, x_8\}$
x_5	$\{x_3, x_4, x_5, x_6, x_7, x_8\}$	$\{x_3, x_5\}$	$\{x_1, x_2, x_3, x_5, x_6\}$	U_1	$\{x_3, x_5\}$
x_6	$\{x_3, x_4, x_5, x_6, x_7, x_8\}$	$\{x_6\}$	$\{x_1, x_2, x_3, x_5, x_6\}$	U_1	$\{x_6\}$
x_7	$\{x_3, x_4, x_5, x_6, x_7, x_8\}$	$\{x_1, x_2, x_4, x_7, x_8\}$	$\{x_4, x_7, x_8\}$	U_1	$\{x_4, x_7, x_8\}$
x_8	$\{x_3, x_4, x_5, x_6, x_7, x_8\}$	$\{x_1, x_2, x_4, x_7, x_8\}$	$\{x_4, x_7, x_8\}$	U_1	$\{x_4, x_7, x_8\}$

Table 4: The partitions based on each fuzzy covering $\mathscr{C}_i$ ($1 \leq i \leq m+1$) and Δ, respectively.

U_1	$\mathscr{C}_1$	$\mathscr{C}_2$	.	.	.	$\mathscr{C}_m$	$\mathscr{C}_{m+1}$	Δ
x_1	P_{1x_1}	P_{2x_1}	.	.	.	P_{mx_1}	$P_{(m+1)x_1}$	$P_{\Delta x_1}$
x_2	P_{1x_2}	P_{2x_2}	.	.	.	P_{mx_2}	$P_{(m+1)x_2}$	$P_{\Delta x_2}$
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
x_n	P_{1x_n}	P_{2x_2}	.	.	.	P_{mx_n}	$P_{(m+1)x_n}$	$P_{\Delta x_n}$

Table 5: The partitions based on each fuzzy covering $\mathscr{C}_i$ $(1 \leq i \leq k-1, k+1 \leq i \leq m)$ and Δ, respectively.

U_1	$\mathscr{C}_1$	$\mathscr{C}_2$	.	.	.	$\mathscr{C}_{k-1}$	$\mathscr{C}_{k+1}$	.	.	.	$\mathscr{C}_m$	Δ
x_1	P_{1x_1}	P_{2x_1}	.	.	.	$P_{(k-1)x_1}$	$P_{(k+1)x_1}$	.	.	.	P_{mx_1}	$P_{\Delta x_1}$
x_2	P_{1x_2}	P_{2x_2}	.	.	.	$P_{(k-1)x_2}$	$P_{(k+1)x_2}$	.	.	.	P_{mx_2}	$P_{\Delta x_2}$
.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.	.
x_n	P_{1x_n}	P_{2x_n}	.	.	.	$P_{(k-1)x_n}$	$P_{(k+1)x_n}$	.	.	.	P_{mx_n}	$P_{\Delta x_n}$

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